Input Market, Partnership and Heterogeneous Innovations

Qinshu Xue Bin Zhao

Click here for the latest version

November 17, 2021

Abstract

This paper studies the relationship between partnership and a firm's innovation strategies and its implications for industrial growth. We empirically documents that forming partnerships across firms is associated with more exploitative (incremental) innovations and less exploratory (radical) innovations. Guided by the result, we propose a tractable growth framework where multi-product firms optimally implement either exploitative or exploratory innovations for each of their product lines, given their partnership status. Although forming a partnership mitigates misallocation by reducing frictions in the input market, it dilutes the overall input market 'vintages' by introducing too many over-developed inputs with limited productivity enhancement and therefore damper the industry growth. Our framework permits analysis on how partnership can affect a firm's innovation strategies and overall industrial growth.

1 Introduction

The input market plays a crucial role in productivity spillovers. An increase in productivity from the upstream can spur productivity gains in its downstream through input linkages. However, as a nature of mass fragmentation of production, the input market (for both intangible or physical intermediate goods) is subject to search friction, which amounts to a considerable amount of costs for firms through two vehicles: (1) the misallocation cost stemmed from imperfect matches with supplies (hence lower productivity gains for the downstream); [1] (2) cost due to higher risk of unsold inventories (hence induce lower price of input and discourage innovations).[2] Confronting with the low quality of matching and the risks of accumulating a high volume of inventories created by the search distortion, a fraction of producers have built up the capability of forming partnerships with others to facilitate an additional channel to sustain a more stable or efficient supply chain. In addition, partnerships can also serve as a possible complementary/catalyst for their productivity growth through persistent innovation efforts.

This new trend in organization structure has sparked many ongoing policy debates centering around whether the economy should offer more relaxed constraints on those integrating activities so that those incorporated firms can obtain more productivity gain. [3] The existing literature thus has cultivated on the following problem: *whether the firms involved in more integrating activities achieve higher productivity by fostering more innovations.* However, this project departure from such scope by looking at the heterogeneity in innovations. For instance, more innovations are not equivalent to more innovations with high "quality" or breakthroughs. Indeed, as argued for long, the contribution of following-up innovations has less impact than those exploratory innovations that bring breakthroughs.[4] Therefore, we are instead asking (1) *whether or not forming a partnership is associated with a specific direction of innovations: more incremental improvements built on existing inventions or more exploratory innovations which can lead to breakthroughs in the*

[1] A large body of literature has discussed the misallocation caused by various distortions sources in intermediate good market: Jones (2013), Bartelme and Gorodnichenko (2014), Fadinger, Ghiglino and Teteryatnikova (2016), Bigio and La'O (2016), Caprettini and Ciccone (2015), Liu (2017), Caliendo, Parro and Tsyvinski (2017), Osotimehin and Popov (2017), and Baqaee and Farhi (2017)

[2] Many works investigate the relationship between price and search frictions in asset markets including investment good, housing and stock market: Bai, Rios-Rull and Storesletten (2012), Lagos, Rocheteau, and Wright, 2014), Piazzesi, Schneider and Stroebel (2013), Ottonello (2019); In addition, another strand of literature in the field of IO also reveals such relationship between search friction and good market: Armstrong and Zhou (2016), Goldberg (1996), Scott-Morton et al. (2001), Busse et al. (2006), Dafny (2010), Gavazza (2016), Joskow (1987), Town and Vistnes (2001) and Salz (2015).

[3] See Economists Column:"The EUs industrial-policy fans want to go back to the 70s": *https://www.economist.com/europe/2018/12/22/the-eus-industrial-policy-fans-want-to-go-back-to-the-70s*

[4] Garcia-Macia, Hsieh & Klenow (2019) argues that in ex-post sense following-up innovations in aggregate contribute more than those exploratory innovations, which is however not contradicted with the argument.

industry; (2) *how those integrated firms can influence input market activities, innovation strategies of other non-integrated entities and the industry as a whole.*

We use data from Factset Resere, Computat, and US Patent and Trademark Office to compile a micro-level dataset on firm-level financial information, partnership, and supplier information, and patent filing for public firms in the US between 2003 and 2016. We further classify the innovation direction by a pattern's citations. A patent is classified as an exploratory(radical) innovation if at least 80% of its citations are based on new knowledge, while a patent is classified as an exploitative(incremental) innovation if at least 80% of its citations are based on old knowledge. Partnerships are associated with more incremental innovation, controlling for firm-level financial characteristics and industry and year fixed effect. On the other hand, it is accompanied by a slowing down innovation progress upon exploratory *R&D*.

We next develop an endogenous growth model where firms choose between exploitative (incremental) and exploratory (radical) innovations for each product line, given their partnership status. A successful exploitative innovation results in an upgrade in the quality of a product line, but such incremental innovation strategies suffer from decreasing return to scale. On the other hand, firms can replace their product lines with a new variety via successful exploratory innovations. There are frictions in the input markets, and we allow products under the partnership to access more efficiency when trading in the input markets. This implies the innovation strategy upon each product is therefore not only affected by the *R&D* efficiency but also is subject to input market environment, of which the degree of friction faced by products depends on whether the product is in a partnership with some other or not.

Our model has several features. First of all, we admit heterogeneous innovation types by extending Klette and Kortum (2004) framework in which only radical innovations are considered. Furthermore, we allow firms to grow in productivity through sourcing, consistent with the literature emphasizing the productivity-enhancing effect by acquiring inputs.[5] The existence of an input market provides incentives for firms to conduct exploitative innovations as they can benefit from trading inputs generated from incremental innovations. Thirdly, our theory accommodates heterogeneity in product efficiency in terms of trading in the input market. Some of the products are born as 'high type' that is feasible to form a partnership with other 'high type' products. Partnerships grant an additional channel of profits and growth, reinforcing the motives to implement incremental innovations on those product lines.

Finally, search externalities are two-folds in our framework. First, market participants

[5]Similar discussions can be found in Pavcnik (2002), Khandelwal and Topalova (2011), Goldberg et al. (2010), Bas and Strauss-Kahn (2015).

cannot coordinate with others to adjust market tightness. Second, the market participants cannot internalize the impact of their innovation policies on the expected 'quality' of input available in the sourcing market. The latter point contributes to suboptimal innovation policies of the firms as they should have implemented relatively more exploratory (less exploitative) innovations if they can internalize such externality. This enables us to refine the policy question on how encouraging more partnerships within industries will affect the innovation policies of non-integrated firms and further affect the growth of the industry.

Innovations differ substantially in their nature: some of them are incremental improvements over the existing technologies, while others constitute radical innovations that can amount to the industry's evolution and introduce creative destruction. The literature initially more concentrated on the latter type. Initiated by Romer (1990), Aghion and Howitt (1992), the endogenous growth model was built on a radical technological change in aggregate representative agent framework. It then has been extended to a firm-level framework as elaborated in Klette and Kortum (2004), Lentz and Mortensen (2008), and recently in Acemoglu et al. (2018). On the other hand, given the greater availability of the micro-level patent dataset, researchers have started exploring incremental innovations' roles. Acemoglu and Cao (2015) extends the Schumpeterian endogenous growth model by allowing incumbents to undertake innovations to improve their products, while entrants engage in more "radical" innovations to replace incumbents. They argue that the growth contributed by incremental innovations can be more important than the radical innovations theoretically but without providing empirical evidence. Garcia-Macia et al. (2019) provides empirical evidence suggesting most growth is driven by incremental innovations. Our model also argues that the value of radical innovation is not manifested immediately but reveals its potential through subsequent developments, which is consistent with Brynjolfsson, Rock and Syverson (2019).

Our work is closely related to the literature on implications of resource misallocation as well. Hsieh and Klenow (2009), Restuccia and Rogerson (2008) are the pioneering efforts that empirically document the significant productivity losses due to resource misallocation. Extending their works, several strands of papers attempt to micro-found the sources of misallocations in various facets. Misallocations due to the financial fraction are developed in Buera et al. (2011) and Midrigan and Xu (2013). An alternative mechanism proposed by Boehm and Oberfield (2019) suggests the misallocation is stemmed from adopting lower quality of input due to weak contract enforcement. David et al. (2016) investigates how imperfect information distorts input sourcing decisions of firms both theoretically and quantitatively. Each of those three arguments agrees on the misallocation due to failure to adopt intermediate goods with high quality, which is highlighted in our model. Our framework aims to complement those works as we further push the role of incomplete information in the implication of resources allocation beyond the distortions in firms' sourcing policies. In our framework, the probability of purchasing inputs that bear 'relatively novel technology' is a result of firms' joint decisions which cannot be internalized by individuals. Such externality affects the future innovation decisions of each firm, distorting the allocations in both input procurement and innovation strategies further.

In addition, our theory adopts the search framework to capture the nature and potential misallocation in acquiring input resources within an industry populated by a large mass of buyers and sellers. There exists a large body of literature discussing search friction in labor input factors and their implication for aggregate welfare since McCall (1970), Diamond (1982), Mortensen (1982), and Pissarides (1990). Recent attentions have been drawn to other production factors, including intermediate good, ideas, and capital. Chiu, Meh, and Wright (2011) examines the allocations of inputs in the form of technology transfer in a frictional exchange market but assume the technology inputs are homogeneous, which implies misallocation is absent. Akcigit et al. (2016) focuses on the patent trading market with search frictions by incorporating additional margin in the mismatch of ideas. They argue that friction contributes to slower growth and loss of welfare and hence suggest the efficiency of the trading market is quantitatively important. A recent work by Ottonello (2019) involves a frictional non-Walrasian physical capital market. The search friction contributes to a lower investment rate at recession since the agents in the market find it optimal to absorb the unemployed capital first rather than build capital on their own. Our framework differs from those works by treating the search friction in the input market with a broader concept covering the trade of both physical (i.e., machine or equipment) and intangible (i.e., ideas or licenses) supplies. Furthermore, the search friction in our model is captured by two-folds of risks rather than one risk channel: (1) risk of failing to meet with sellers due to mass amount of market participants; (2) risk of meeting a seller with "low quality" input due to imperfect information prior to a meeting.

Finally, our work relates to organizational structure and its implication for innovation incentives. Bena and Li (2014) documents the empirical facts that acquirers with prior technological linkage to their target firms produce more patents afterwards. Phillips and Zhdanov (2013) further develops theory and empirical findings that small firms have greater incentives to innovate and prefer being acquired by large firms which strategically choose to avoid *R&D* races against small firms. Other than vertical integration and its impact on the quantity of innovations, Jansen et al. (2006) investigates how the organizational structure within a firm can influence the innovation outcome in both exploitative and exploratory margins. Their results indicate that formal coordination across the units within firm negatively affects exploratory innovation and positively influences exploitative innovation. Our model allows for a broader notion of integrating relationships and focuses on how such integrating activities affect the innovation directions of firms. Our implication is consistent with Bena and Li (2014), Phillips and Zhdanov (2013) in the exploitative margin and aligns with intuitions in Jansen et al. (2006) but with notions of organizational structure across firms instead of within firms.

The rest of the paper is organized as follows: we present motivating empirical facts in Section 2 with data description and empirical strategy; Section 3 scratches a benchmark model which starts from a firm-level problem in Section 3.1. From Section 3.2-3.7, we break down the firms' problem into product line levels and characterize the innovation direction decision-making. In Section 3.8, we close the economy and characterize the balanced growth path. All related proof can be found in the Appendix.

2 Motivating Empirical Facts

We attempt to answer our first question in an empirical content: whether or not forming partnership is associated with particular patterns in those firms' innovation strategies. Three datasets are used in documenting supply chain relationship including partnering information and patents record at firm-level.

Factset Revere: The first source is Factset Revere Supply Chain Relationships Data, which covers the period from 2003 to 2016 and 13,000 US private and public firms. For each documented firm, we would observe comprehensive firm's network structure with specific relationships including supply-chain relationship, equity-holding relationship and other strategic relationships, etc. An example of firm's production network is illustrated graphically below: [6]

Figure 5: Boeing's Relationships Network

As shown above, beyond standard roles within a supply chain: customers-suppliers (for physical goods, i.e. raw materials, machines), licensees-licensors (for intangible goods, i.e. software), the data documents those roles associated with control rights of firms including joint venture, equity holder, research collaboration, etc. We treat those relationship with such nature as partnership, which allows us to further identify the firms involving in partnership.

[6] The image is provided in FactSet Data Guidelines as an example that shows the richness of modern production networks.

Compustat North American Fundamentals (Annual): Another source documenting firm-level data is Compustat which covers public firms in the United States since 1976. It reports firm-level sales, employment data with which is matched with the supply-chain data as our major controls in the empirical analysis.
United States Patent and Trademark Office: The last source is USPTO dataset which reports detailed information upon utility patents granted. In particular, it includes the citations information of a given patent, based on which we construct two measures of innovation types:

- a patent is classified as an exploratory innovation if at least 80% of its citations are based on new knowledge; [7]
- a patent is classified as an exploitative innovation if at least 80% of its citations are based on existing knowledge. [8]

We match the two patent measures with our firm-level data gathered from the other two database.

2.1 Empirical Strategy

Given the data availability, we identify the set of partners of a given firm i, denoted by $S_{partner}(i)$:

$$S_{partner}(i) = S_{shareholder}(i) \cup S_{shareholding}(i) \cup S_{JV_{partner}}(i) \\ \cup S_{researchcollaboration}(i) \cup S_{poolinglicence_{partner}}(i)$$

To examine the correlation between partnership and innovation direction, we construct a new variable named as density of partnership $d_{x-partner}(i)$, which captures the fraction of partners in a particular supply-chain relationship $x \in$ {supplier, customer, licensor, licensee} with a given firm i, [9] where **suppliers provide physical input to a given firm i, and licensors provide intangible intermediate goods to firm i**:

$$d_{x-partner}(i) = \frac{\#(S_x(i) \cap S_{partner}(i))}{\#S_x(i)}, x \in X \equiv \{\text{supplier, customer, licensor, licensee}\}^{10}$$

This measure ignores the quantity transacted per supply-chain relationship, which is unobserved. We specify the OLS as following:

$$y_{it} = \beta_0 + \sum_{j \in J} \beta_j log(d_{j-partnert-1} + 1) + B * Z_{it-1} + \epsilon_{it}$$

[7] New knowledge is defined as the citations are not recorded in the firm's other patents nor the associated citations.
[8] Existing knowledge is defined as the citations are recorded in the firm's other patents or the associated citations.
[9] Summary Statics is available in Appendix B
[10] $\{x\}$ are not exclusive sets in the sense that we may encounter the case where the supplier of firm i is also a licensor of firm i

where $\{y_{it}\}$ are the two fractions of innovations of firm i at time t: that are exploitative and exploratory, and Z_{it-1} are the controls including: employment size (size of firms), SIC2 (manufacturer), # patents at $t-1$, # citations per patent, and number of total partners of firm i at time $t-1$.

2.2 Brief Results

As illustrated by the OLS regression results, it appears that the density of partnership is related to rigidity in incremental innovations. In particular, a greater density of partnership in both intangible goods and physical suppliers is associated with more incentives to conduct exploitative (incremental) innovations. On the other hand, it is accompanied by a slowing down innovation progress upon exploratory *R&D*. [11] Furthermore, the firm's size appears to be negatively correlated with growth in exploratory innovation, which is consistent with the finding in Acemoglu et al. (2018). Apart from the partnership, there exists a strong positive relationship between the number of patents and exploratory innovations. It is consistent with our presumption that exploitative innovation experiences decreasing return to scale to some extent so that firms will eventually conduct exploratory innovations after sufficient amounts of incremental innovations. Unsurprisingly, the quality of patent is associated with more exploitative innovations, indicating that firms' incentive to conduct follow-up innovations depends on its associated return.

Variable	Obs	Mean	Std. Dev.	Min	Max
den_lic_partner	11,896	0.295	0.355	0	1
den_licee_partner	11,323	0.318	0.350	0	1
den_cc_partner	40,623	0.183	0.295	0	1
den_sup_partner	27,083	0.238	0.310	0	1

OLS Regression Result:

[11] We report the empirical results both with and without controlling the density of partnership in 'pure suppliers'

VARIABLES	(1) Exploratory	(2) Exploratory	(3) Exploitative	(4) Exploitative
log_density of partners among suppliers	0.003	0.001	0.020*	0.020*
	(0.013)	(0.012)	(0.012)	(0.012)
log_density of partners among licensors	-0.030*	-0.030*	0.034**	0.033**
	(0.016)	(0.016)	(0.015)	(0.016)
log_density of partners among customers	-0.010	-0.008	0.004	0.004
	(0.009)	(0.009)	(0.007)	(0.007)
log_density of partners among licensees	0.022		0.008	
	(0.018)		(0.016)	
log_number of total partners	0.011	0.012	-0.006	-0.006
	(0.010)	(0.010)	(0.007)	(0.007)
log_number of patents	0.114***	0.113***	0.107***	0.107***
	(0.014)	(0.014)	(0.015)	(0.015)
log_number of citations per patent	-0.004	-0.004	0.039**	0.039**
	(0.017)	(0.017)	(0.017)	(0.017)
log_employment	-0.021***	-0.021***	0.011*	0.011*
	(0.008)	(0.008)	(0.006)	(0.006)
sic2	-	-	-	-
Constant	0.045	0.032	0.050	0.045
	(0.059)	(0.059)	(0.045)	(0.045)
Observations	959	959	959	959
R-squared	0.267	0.266	0.333	0.333
Adjusted/Pseudo R-Square	0.245	0.245	0.199	0.199

Robust standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Alternative OLS Specification with controlling partner density of pure suppliers

VARIABLES	(1) Exploratory	(2) Exploratory	(3) Exploitative	(4) Exploitative
log_den_sup	-0.006	-0.006	0.045**	0.045**
	(0.020)	(0.020)	(0.019)	(0.019)
log_den_lic	-0.030*	-0.030*	0.033**	0.033**
	(0.018)	(0.018)	(0.016)	(0.016)
log_den_cc	-0.020**	-0.020**	0.014*	0.014*
	(0.009)	(0.009)	(0.007)	(0.007)
log_den_licee	0.005		0.004	
	(0.020)		(0.019)	
lnpat_per_firm	0.113***	0.113***	0.108***	0.107***
	(0.014)	(0.014)	(0.015)	(0.015)
lnciteperpat	-0.003	-0.003	0.039**	0.039**
	(0.017)	(0.017)	(0.017)	(0.017)
log_emp	-0.021***	-0.021***	0.011*	0.011*
	(0.007)	(0.007)	(0.006)	(0.006)
log_den_pure_sup	0.015	0.015	-0.021	-0.021
	(0.020)	(0.020)	(0.019)	(0.019)
sic2				
Constant	0.078*	0.078*	0.015	0.015
	(0.046)	(0.046)	(0.041)	(0.041)
Observations	959	959	959	959
R-squared	0.267	0.267	0.335	0.335
Adjusted/Pseudo R-Square	0.245	0.244	0.199	0.199

Robust standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

3 Model

Time is continuous. The market is populated by a fixed unit measure of differentiated final goods indexed by $j \in \mathcal{J} \equiv [0,1]$. Each product is produced by a single, risk neutral firm indexed by $f \in \mathcal{F}$, and firms are, in principal, multi-product producers. Let $M < 1$ denote by the aggregate measure of firms, which is also fixed in the benchmark. There exists three departments within each firm: *R&D* sector, intermediate supplies sector, and final good sector, which further constitute teams at product line level. To produce a given final good j, a firm must utilize technique z_j which is spurred from its R&D team or is enhanced by purchasing parts from other firms. For simplicity, the profit generated from producing the product with z_j is in linear form with

$$\pi(z_j) = \pi z_j$$

where π is a constant across product space. The innovation schedule conducted by firms is not necessarily solely climbing along the technology ladder. That is, firm can direct the *R&D* sector to either incrementally develop over the current technology for producing the product or implement exploratory innovation to replace the developed product line. Let $\theta_j \in \Theta \equiv \{\theta_{j,D}, \theta_{j,E}\}$ denote by the exploitative (incremental) innovation ($\theta_{j,D}$) or exploratory (radical) innovation ($\theta_{j,E}$) directed in good j. There is no absorbing state in innovation types: a product currently produced under θ_D can be replaced by a successful exploratory innovation θ_E while a technique θ_E can be developed further by firm to be θ_D. For tractability, we are abstract from endogenizing *R&D* efforts, and focus on firms' decision on the timing of switching innovation directions. That is, we are shaping the model as an optimal stopping problem. The 'later' the firm decides to direct to exploratory innovation on a given product line, the more incremental innovations that have been taken relative to implementing one exploratory innovations.
Assumption 1: Innovation arrival rates of both types are constant across firms and time, and *R&D* is costless. [12]

3.1 θ_D Exploitative (Incremental) Innovation, Final good usage and Input usage

Conditional on directing the *R&D* sector to conduct incremental innovation on a given product j, the outcome after the *R&D* efforts realizes in a random fashion which comes in two folds: (1) an improvement of current technology level arrives at a Poisson rate i; (2) the degree of improvement/size of improvement step, $S_{n_j}(x_j, \bar{z})$, which is composed of number of incremental innovations that has been taken upon the product, n_j; a random variable

[12] As discussed in many relevant literature, constant *R&D* efforts per product line is sufficient to generate either constant or decreasing *R&D* intensity in firm sizes. See Klette and Kortum (2004), Akcigit and Kerr (2016) and Acemoglu et al. (2018). In terms of the cost of R&D, one can think of the constant term *pi* in our profit function as it has already incorporated the associated R&D costs. We will explicitly endogenize the R&D efforts in the future extension.

capturing the compatibility of the innovation, x_j, drawn from a uniform distribution with support $[0,1]$ and the average technology level $\bar{z}$ in the market:

$$S_{n_j} = S a^{n_j} x_j \bar{z}, \;\; a \in (0,1)$$

Conditional on successful innovation and a realization of compatibility degree x_j, the firm decides where to apply the newly obtained knowledge/technique: either to intermediate supplies sector or to the final good sector. If the technique is applied in final good sector, the step size coefficient S is the specified to be a constant η. The law of motion of technology level adopted for producing the good follows:

$$z'_j = z_j + \eta_{n_j}(x_j, \bar{z}) = z_j + \eta a^{n_j} x_j \bar{z}$$

If adopted in intermediate sector, a 'part' will be produced and it can be traded in frictional input market and simultaneously used in final good sector but with a discounted contribution at step size $\lambda_n(x, \bar{z})$:

$$\lambda_{n_j}(x_j, \bar{z}) = \lambda a^{n_j} x_j \bar{z}, \;\; \lambda \leq \eta$$

To remind, upon a successful θ_D innovation, the state of innovation index n_j of the product j evolves:

$$n'_j = n_j + 1, \;\; \textit{if incremental innovation arrives}$$

Nevertheless, the productivity gains from sourcing does not accumulate the index as it is not due to successful incremental innovations. The law of motion of technology level of product j produced by firm f if adopting a part k produced by some other firm g is given by:

$$z'_j = \begin{cases} z_j + \lambda a^{n_k} x_{kj} \bar{z} & \textit{if adopting part } k \textit{ produced by firm } g \textit{ without partnership} \\ z_j + \eta a^{n_k} x_{kj} \bar{z} & \textit{if adopting part } k \textit{ produced by firm } g \textit{ under partnership} \end{cases}$$

where n_k is the number of incremental innovation has been adopted associated with the traded part, and x_{kj} is the compatibility of the part to the buyer-firm drawn from the uniform distribution, which is a independent draw rather than x_k, the compatibility level to good k itself. Furthermore, we allow firms to trade under partnership, which grants more specifications bonus with higher productivity gain coefficient, η.

3.2 Heterogeneous Types in Product Lines & Partnership

We assume there exists two types at product-level, $\phi \in \{H, L\}$. A high type (H) product line has the capability of forming partnership with others when sourcing while a low type (L) product line cannot. In the benchmark model, we are abstract from the scenario where input market is pooled with all product lines. Instead we are assuming two separate input markets: (1) common (Type L) input market; and (2) Partner (Type H) input market.

To be specific, there exists probability α that a high type product line can sell and access to H type market, **in addition to L market**. Once it enters the H market, regardless of usage of its innovation to the owner, innovations will be sold to the market. The type of product line is not absolutely invariant. A successful exploratory innovation resets not only the innovation index n to 0 but also the types. The probability of drawing a type H assigned to the renewed product line is fixed at some $h \in [0, 1)$.

3.3 Trade in Input Markets

As pointed out previously, a product line can be benefited from sourcing part produced by others' technique when the direction of innovation operated is θ_D. The trade of parts is organized in separate input markets, each of which is a two-sided market. The common input market (L market) is participated by all products (not all firms) whose R&D teams are currently implementing θ_D innovations while the partner input market only grant accessibility to H type product lines. Both input markets are subject to search friction. That is, the probability of meeting a counter party in the market for any participants is less than 1. To be more specific, such probability is governed by the ratio of sellers to the buyers in the market. Given the feature, it is likely that a firm will take long time to sell its input. Before a successful sale, a firm may have already hold multiple inputs to put on the sourcing market. Hence for simplicity, we abstract from tracking down the inventory history of any given part by assuming there exists a complete financial sector populated by a continuum of capitalists/bankers who are able to provide full insurance covering the risk of inventory. Alternatively speaking, it is optimal for each input seller to signs a collateral contract with a banker to borrow cash out in advance. As a return, the option value of the input sale is then transferred to banker: any payment from selling the part will be transferred to the banker.

Assumption 2: There exists a complete financial market, providing full insurance coverage on selling inputs.

Given the assumption, for any product line j manger (including both types) turning her knowledge into part, it is optimal for her to sell the option of selling part j to the financial sector and obtain transfer at $T(n_j, \bar{z})$, where we have assumed the number of incremental innovations that have been taken for the product j and the aggregate technology level and the matching efficiency are observable information to bankers. After doing so, she hires a sales agent to enter the input market governed by standard Cobb-Douglas matching function $m(n_a, n_b)$:

$$m^{\phi}(y_a, y_b) = \zeta y_a^{\phi^{\varepsilon}} y_b^{\phi^{1-\varepsilon}}$$

where y_a^{ϕ} is the measure of sales agents in the input market $\{\phi\}$ and y_b^{ϕ} is the measure of the buyers. Upon a successful meeting, the compatibility of the part to the buyer then is realized. Given the realization, the buyer determines whether to purchase it or not. Given the matching function, it immediately follows that the probability that a sales agent meets with a buyer reads as $m_a^{\phi} \equiv \zeta(\frac{y_b^{\phi}}{y_a^{\phi}})^{1-\varepsilon}$. Similarly, let the probability that a buyer meets

with a sales agent denote by $m_b^\phi \equiv \zeta(\frac{y_a^\phi}{y_b^\phi})^\varepsilon$, $\phi \in \{H, L\}$. To highlight, prior to a successful meeting with a sales agent, contrast by the information structure under the transaction between firms and bankers, we are assuming buyers cannot observe the innovation index n. In other words, there exists no sub-market for each n to direct search behavior.
Remark 1: Input market is not segmented by innovation index n.
This construction plays a crucial role in the economy since it creates externality induced by firm's innovation strategy. Intuitively speaking, imagine a scenario where all product managers decide to delay their θ_E innovations and implement more incremental innovations. Such product-level decision will induce a greater average of innovation index for the input market, $\mathbb{E}_{\{\mu_{n_k}^\phi\}}[n_k]$,[13] which in turns lower the expected return on sourcing. Given the large mass of entity in the industry, each firm make innovation decisions individually and have no measures to coordinate, which leave a risk of creating market failure.

3.4 θ_E Exploratory Innovation and Product Life-Cycle

Conditional on exercising exploratory innovation, firms direct their *R&D* team of a given product to work on a new product line which will replace the current one. Deviate from Klette & Kortum (2004) in which a successful innovation in new product line destroys randomly an existing product line k produced with θ_D technique at some technology level z_k and replace it with new product with a technology improvement, we take a simpler notion of θ_E innovation: *the exploratory innovations results in a direct replacement of the current product line.*

$$z' = z + \chi\bar{z}$$

Assumption 4: The exploratory innovation does not contribute to technology improvement directly. That is, $\chi = 0$
This notion is align with current discussion by Brynjolfsson, Rock and Syverson (2019), where they find a drastic development on general purpose technology tends to not manifest its productivity advent until it becomes mature. Let ν denote the arrival rate of exploratory innovation. Upon a successful exploratory innovation on any given product line j, it is renewed in the sense that the number of incremental innovation is reset, $n_j = 0$. To remind, when implementing θ_E innovation on a given product, the product line cannot benefit from sourcing, and the realized innovation cannot be transformed for input-usage by construction.

3.5 Firm's Problem

To remind, we assume the profit function of a product line j is linear in technology z_j:

$$\pi(z_j) = \pi z_j$$

[13] Let $\{\mu_{n_k}^\phi\}$ be the distribution of index n in market ϕ.

where π is constant across all product lines. Now we first summarize the state variables of the problem. Let $\boldsymbol{z_f}$ be the vector of technology level of the product lines operated by firm f: $\boldsymbol{z_f} = \{z_{f,j_1}, ..., z_{f,j_{mf}}\}$; similarly define $\boldsymbol{n_f} = \{n_{f,j_1}, ..., n_{f,j_{mf}}\}$ as the vector of number of incremental innovations taken over the product lines operated by firm f; finally let $\boldsymbol{\phi_f} = \{\phi_{f,j_1}, ..., \phi_{f,j_{mf}}\}$, where m_f be the cardinally of the product line space owned by firm f. To suppress the notation, we further define $\boldsymbol{S_f} \equiv \{\boldsymbol{z_f}, \boldsymbol{n_f}, \boldsymbol{\phi_f}\}$. The value function of a given firm f is characterized by:

$$
(1)\quad rW(\boldsymbol{S_f}) - \dot{W}(\boldsymbol{S_f}) = \left\{ \begin{array}{c}
\sum_{o\in M_L^f} \Big\{ \pi z_{f,j_o} + \max_{\{\theta_{D_{j_o}}, \theta_{E_{j_o}}\}} \Big\{ \Big\{ i \cdot \Big[\int_{x_s^L}^1 \Big[W\Big(\boldsymbol{S_f} \backslash \{z_{j_o}, n_{j_o}\} \cup \{z_{j_o} + \eta_{n_{j_o}(x,\bar{z})}, n_{j_o}+1\} \Big) - W(\boldsymbol{S_f}) \Big] dx \\
+ \int_0^{x_s^L} \Big[W\Big(\boldsymbol{S_f} \backslash \{z_{j_o}, n_{j_o}\} \cup \{z_{j_o} + \lambda_{n_{j_o}(x,\bar{z})}, n_{j_o}+1\} \Big) - W(\boldsymbol{S_f}) + T_L(n_{j_o}, \bar{z}) \Big] dx \Big\} \\
+ m_b^L \mathbb{E}_{\{\mu_{n_k}^L\}} \Big\{ \int_{x_b^L}^1 \Big[[W\Big(\boldsymbol{S_f} \backslash \{z_{j_o}\} \cup \{z_{j_o} + \lambda_{n_k(x_{kj_0}, \bar{z})}\} \Big) - P_L(x_{kj_o}, n_k, \bar{z}) - W(\boldsymbol{S_f}) \Big] dx_{x_{kj_o}}; \\
\Big\{ \nu \Big\{ h \Big[W\Big(\boldsymbol{S_f} \backslash \{n_{j_0}, L_{j_o}\} \cup \{0, H_{j_o}\} \Big) - W(\boldsymbol{S_f}) \Big] + (1-h) \Big[W\Big(\boldsymbol{S_f} \backslash \{n_{j_0}, L_{j_o}\} \cup \{0, L_{j_o}\} \Big) - W(\boldsymbol{S_f}) \Big] \Big\} \Big\} \\
+ \sum_{o\in M_H^f} \Big\{ \pi z_{f,j_o} + \max_{\{\theta_{D_{j_o}}, \theta_{E_{j_0}}\}} \Big\{ \Big\{ i \cdot \Big[\int_{x_s^H}^1 \Big[W\Big(\boldsymbol{S_f} \backslash \{z_{j_o}, n_{j_o}\} \cup \{z_{j_o} + \eta_{n_{j_o}(x,\bar{z})}, n_{j_o}+1\} \Big) - W(\boldsymbol{S_f}) \Big] dx \\
+ \int_0^{x_s^H} \Big[W\Big(\boldsymbol{S_f} \backslash \{z_{j_o}, n_{j_o}\} \cup \{z_{j_o} + \lambda_{n_{j_o}(x,\bar{z})}, n_{j_o}+1\} \Big) - W(\boldsymbol{S_f}) + T_L(n_{j_o}, \bar{z}) \Big] dx + \alpha \Big(T_H(n_{j_o}, \bar{z}) - F_H(n_{j_o}, \bar{z}) \Big) \Big\} \\
+ m_b^L \mathbb{E}_{\{\mu_{n_k}^L\}} \Big\{ \int_{x_b^L}^1 \Big[[W\Big(\boldsymbol{S_f} \backslash \{z_{j_o}\} \cup \{z_{j_o} + \lambda_{n_k(x_{kj_0}, \bar{z})}\} \Big) - P_L(x_{kj_o}, n_k, \bar{z}) - W(\boldsymbol{S_f}) \Big] dx_{kj_o} \\
+ m_b^H \mathbb{E}_{\{\mu_{n_k}^H\}} \Big\{ \int_{x_b^H}^1 \Big[[W\Big(\boldsymbol{S_f} \backslash \{z_{j_o}\} \cup \{z_{j_o} + \eta_{n_k(x_{kj_0}, \bar{z})}\} \Big) - P_H(x_{kj_o}, n_k, \bar{z}) - W(\boldsymbol{S_f}) \Big] dx_{kj_o}; \\
\Big\{ \nu \Big\{ h \Big[W\Big(\boldsymbol{S_f} \backslash \{n_{j_0}, H_{j_o}\} \cup \{0, H_{j_o}\} \Big) - W(\boldsymbol{S_f}) \Big] + (1-h) \Big[W\Big(\boldsymbol{S_f} \backslash \{n_{j_0}, H_{j_o}\} \cup \{0, L_{j_o}\} \Big) - W(\boldsymbol{S_f}) \Big] \Big\} \Big\}
\end{array} \right\}
$$

where $M_L^f \equiv \{j_o : \phi_{j_o} = L \text{ and } \phi_{j_o} \in \boldsymbol{\phi_f}\}$, $M_H^f \equiv \{j_o : \phi_{j_o} = H \text{ and } \phi_{j_o} \in \boldsymbol{\phi_f}\}$. Also note that we have a set of cutoffs $\{x_s^L, x_b^L, x_s^H, x_b^H\}$. x_s^ϕ captures the lower bound of compatibility of innovation to be utilized for final good specification if the product type is ϕ. Similarly, x_b^ϕ captures the lower bound of compatibility of parts to be purchased in input market with type ϕ. Notice that the cutoffs are independent from product index because the compatibility degree is an i.i.d draw for each product. We briefly interpret the terms on the right hand side which captures the net inflow of value of managing the product profiles. First notice that the value flow can be decomposed into two exclusive blocks which represents the value flow contributed by the product lines managed by firm f that are type L: M_L^f and that contributed by those product lines that are type L: M_H^f. Observe that within each block, firms are choosing optimal innovation direction between θ_E and θ_D for each product line in the set. If the firm f directs to θ_D innovation for project j_o in block M_L^f (M_H^f), then it generates value flow captured by the red (orange) terms which are further composed of the value-added due to successful *R&D* and that due to the gains from sourcing activity in input market. On the other hand, if directing for θ_E innovation, it generates value from resetting the product line innovation index to zero as shown in blue (green) terms. Obviously, the optimal decision is to choose the innovation direction that gives more values to the product j_o given its current state. Thanks to the independence of optimal choice in innovation direction for each product line, the firm's problem can be boiled down to product line level:

Lemma 1 (linearity): $W(\boldsymbol{S_f}) = \sum_{m=1}^{m_f} V_{\phi_{j_m}}(z_{j_m}, n_{j_m}, \bar{z})$

Proof: *See Appendix A.*

Furthermore, owing to the heterogeneity in product lines, we further classify the problems into two sets of problems, namely, the high type product line and low type product line problem. We suppress the subscript of product line index j and its owner f in the following analysis.

3.6 Value Functions of Product line $(z, n, \bar{z})$ with Type $\phi = L$

Given the aggregate productivity level $\bar{z}$, the value function of a product line whose current technology level is z with index of exploitative progress at n is characterized by the following HJB equation:

$$(2)\ rV_L(z,n,\bar{z}) - \dot{V}_L(z,n,\bar{z}) = \left\{ \begin{array}{c} \pi z + \max_{\{\theta_D,\theta_E\}} \Bigg\{ i \Bigg\{ \int_{x_s^L}^1 \left[V_L(z+\eta a^n x\bar{z}, n+1, \bar{z}) - V_L(z,n,\bar{z})\right] dx \\ + \int_0^{x_s^L} \left[V_L(z+\lambda a^n x\bar{z}, n+1, \bar{z}) - V_L(z,n,\bar{z}) + T_L(n,\bar{z})\right] dx \Bigg\} \\ + m_b^L \mathbb{E}_{\{\mu_{n_k}^L\}} \Bigg\{ \int_{x_b^L}^1 \left[V_L(z+\lambda a^{n_k} x\bar{z}, n, \bar{z}) - V_L(z,n,\bar{z}) - P_L(x,n_k,\bar{z})\right] dx \Bigg\}; \\ \nu \Bigg[hV_H(z,0,\bar{z}) + (1-h)V_L(z,0,\bar{z}) - V_L(z,n,\bar{z}) \Bigg] \Bigg\} \end{array} \right\}$$

The product line problem is written in the same fashion as shown in the firm's problem. To shed some lights on the solution of the value function, it appears to be easy for us to presume some specific form of certain terms in the equations: (1) we presume that $T_L(n,\bar{z})$ can be written in the linear form:

$$T_L(n,\bar{z}) = a^n t_L \bar{z}$$

where a^n indicates the discount of part's value with index n; t_L captures the efficiency of the common input market L. It turns out that the presumption holds when we close the economy in the later sections. Given the presumption, we can characterize the value flow attributing to the input market. As standard in the DMP search literature, upon a successful meeting, information including $\{x, z, n_k, \bar{z}\}$ are revealed to two parties. Given the information, a sales agent (seller) holding input k with index n_k and the corresponding buyer bargain over the transfer $P_L(x, n_k, \bar{z})$ through Nash-Bargaining:

$$\max_P \left[[V_L(z+\lambda a^{n_k} x\bar{z}, n, \bar{z}) - V_L(z,n,\bar{z})] - P \right]^{1-\omega_L} \left[P - G_L(n_k,\bar{z}) \right]^{\omega_L}$$

where ω_L is the bargaining power hold by the sales agent, and $G_L(n_k,\bar{z})$ denotes the value of keep selling input indexed by n_k. If the transaction is successful, the agent will retire immediately. There exists free entry condition imposed on financial sector, which implies that we must have the value of selling the parts equivalent to the collateral borrowing:

$$G_L(n_k,\bar{z}) = T_L(n_k,\bar{z})$$

Solving the bargaining problem, one can obtain the price of input with index n_j:

$$P(x, n_k, \bar{z}) = \omega_L \underbrace{[V_L(z + \lambda a^{n_k} x\bar{z}, n, \bar{z}) - V_L(z, n, \bar{z})]}_{\text{Value-added of the part } k \text{ to the buyer}} + (1 - \omega_L) \qquad \underbrace{T(n_k, \bar{z})}_{\text{value of holding the part}}$$

Therefore, the price of the input at state $(x, n_k, \bar{z})$ is a weighted average of value-added to product-line if purchasing and the option value of selling the part.

Given the presumption and owing to the contractility by Klette & Kortum (2004), the solution to the problem (2) turns out to follow the linearity property in the solution to the problem (2), which is summarized by the following Lemma:

Lemma 2: The solution to (2) *is given by:*

$$V_L(z, n, \bar{z}) = Az + B_L(n)\bar{z}, \ \text{for } n \in \{0, 1, 2, 3, ..., n_L^*\}$$

where

$$A = \frac{\pi}{r},$$

$$(r - g)B_L(n) = \begin{cases} i[a^n C_L + B_L(n+1) - B_L(n)] + m_b^L D_L & \text{if } n < n_L^* \\ \nu[hB_H(0) + (1-h)B_L(0) - B_L(n)] & \text{if } n \geq n_L^* \end{cases}$$

with

$$C_L = A\left(\eta \frac{1 - {x_s^L}^2}{2} + \lambda \frac{{x_s^L}^2}{2}\right) + x_s^L t_L,$$

$$D_L = (1 - \omega)\mathbb{E}_{\{\mu_{n_k}^L\}}[a^{n_k}]\left[A\lambda \frac{1 - {x_b^L}^2}{2} - (1 - x_b^L)t_L\right],$$

$$n_L^* = \min\left\{n \in \mathbb{Z} \cup \{0\} : i[a^n C_L + B_L(n+1) - B_L(n)] + m_b^L D_L \leq \nu[hB_H(0) + (1-h)B_L(0) - B_L(n)]\right\}.$$

Proof: *See Appendix A.*

The solution is consistent with the intuition. There exists a maximal degree of exploitative effort for each product line. Upon reaching such threshold, firms direct the research team to exploratory innovation to renew the incremental innovation index. This sort of problem is classified to optimal stopping problem. The benefit from implementing θ_D development on a given product line is decreasing in the number of successful exercises in θ_D innovation, which is illustrated by the first line of solution to $B(n)$. Recall that the gain from implementing θ_D innovation comes from two-fold: productivity improvement and gains from trade in input-market. The solution also reflects the point: the level of $\{B(n)\}$ is increasing in the contribution from successful *R&D* which is indicated by $a^n C_L$ and from sourcing from common input market (L), which is captured by $m_b^L D_L$. Obviously, the growth in value coming from exploitative innovation is diminishing as n accumulates. Furthermore, note that $m_b^L D_L$ is not from n. Indeed as we have shown in the Appendix, the sourcing contribution $m_b^L D_L$ diminishes to 0 if both n_L^* and n_H^* goes to infinity. This implies the value flow of conducting θ_D innovation converges to null, which further indicates the exploratory innovation will be favored eventually. Furthermore, notice that for each product line manager, she cannot internalize the impact of joint switching strategies of all other product lines on the average innovation index of the common sourcing market. Otherwise, the outside option will be shifted up, hence the switching point should be lower. Finally notice that we allow for $n^* = 0$, which means no θ_D innovation will be implemented at all but only focus on θ_E innovation, of which cases will be root out in the later sections.

3.7 Value Functions of Product line $(z, n, \bar{z})$ with Type $\phi = H$

Similar to the product line problem with type L, the corresponding HJB equation for type H product line is given by:

$$(3)\ rV_H(z,n,\bar{z}) - \dot{V}_H(z,n,\bar{z}) = \left\{ \begin{array}{c} \pi z + \max_{\{\theta_D,\theta_E\}} \Bigg\{ i \Big\{ \int_{x_s^H}^1 \left[V_H(z+\eta a^n x\bar{z}, n+1, \bar{z}) - V_H(z,n,\bar{z})\right] dx \\ + \int_0^{x_s^H} \left[V_H(z+\lambda a^n x\bar{z}, n+1, \bar{z}) - V_H(z,n,\bar{z}) + T_L(n,\bar{z})\right] dx + \alpha\Big(T_H(n,\bar{z}) - F_H(n,\bar{z})\Big)\Big\} \\ + m_b^L \mathbb{E}_{\{\mu_{n_k}^L\}} \Big\{ \int_{x_b^L}^1 \left[V_H(z+\lambda a^{n_k} x\bar{z}, n, \bar{z}) - V_H(z,n,\bar{z}) - P_L(x,n_k,\bar{z})\right] dx \Big\} \\ + \alpha m_b^H \mathbb{E}_{\{\mu_{n_k}^H\}} \Big\{ \int_{x_b^H}^1 \left[V_H(z+\eta a^{n_k} x\bar{z}, n, \bar{z}) - V_H(z,n,\bar{z}) - P_H(x,n_k,\bar{z})\right] dx \Big\}; \\ \nu \Big[hV_H(z,0,\bar{z}) + (1-h)V_L(z,0,\bar{z}) - V_H(z,n,\bar{z}) \Big] \Bigg\} \end{array} \right\}$$

What distinct from the low-type product problem is high type product line can benefit from participating partnership with probability α. If the partnership opportunity comes, the product manager is able to sell to the partner input market (H), which worth at $T_H(n, \bar{z})$, after paying a fixed cost to manage the partnership relation at $F_H(n, \bar{z})$. Throughout the benchmark sections, we are abstract from gains from selling parts to partners by assuming $F_H(n, \bar{z}) = T_H(n, \bar{z})$. Furthermore, she will enter the market to source from partner with greater input complementarity captured by the coefficient η. Given the additional channel of value inflow, it is easy to see that given the same state except for the type, $(z, n, \bar{z})$, high type product line generates greater value stream than the low type does. As done previously, we presume the solution to option value of selling part k in partner input market, (H) is linear:

$$T_H(n_k, \bar{z}) = a^{n_k} t_H \bar{z}$$

As we will show later, the high type input market is facing a greater market tightness since all participants will provide their innovation to others as input for others. The following lemma summarizes the solution to the problem:

Lemma 3: The solution to (3) *is given by:*

$$V_H(z, n, \bar{z}) = Az + B_H(n)\bar{z}, \ \textit{for } n \in \{0, 1, 2, 3, ..., n_H^*\}$$

where

$$A = \frac{\pi}{r},$$

$$(r-g)B_H(n) = \begin{cases} i[a^n C_L + B_H(n+1) - B_H(n)] + m_b^L D_L + m_b^H D_H & \text{if } n < n_H^* \\ \nu[hB_H(0) + (1-h)B_L(0) - B_H(n)] & \text{if } n \geq n_H^* \end{cases}$$

with

$$D_H = \alpha(1-\omega)\mathbb{E}_{\{\mu_{n_k}^H\}}[a^{n_k}]\left[A\eta\frac{1-x_b^{H\,2}}{2} - (1-x_b^H)t_H\right],$$

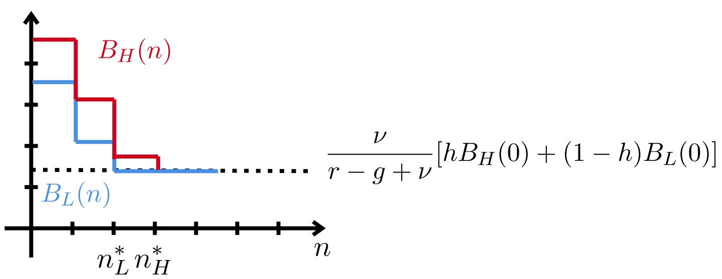
$$n_H^* = \min\Big\{n \in \mathbb{Z}\cup\{0\} : i[a^n C_L + B_H(n+1) - B_H(n)] + m_b^L D_L + m_b^H D_H \leq \nu[hB_H(0) + (1-h)B_L(0) - B_H(n)]\Big\}.$$

Proof: *See Appendix A.*

The solution is consistent with the intuition. It is apparent to see that the return on choosing exploitative innovation is strictly higher than that of low type product line due to the extra channel of value inflow at each innovation index n. Together with the fact that all product lines are confronting with the randomness coming from θ_E innovation upon the renewed product line's type. The return on implementing θ_E for both types are homogeneous, which implies high type product lines have greater incentives to cultivate further on their existing products. We summarize such heterogeneity in innovation decisions across types of product lines in the following claim:

Proposition 1: *A product line involving with partnership tends to implement more incremental innovations, that is, $n_H^* \geq n_L^*$.*

Proof: *See Appendix A.*

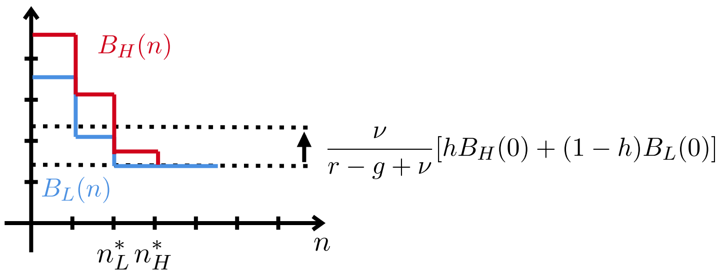


The intuition based on the comparison upon growth channels and opportunity cost also offer insights in comparative statics. For instance, imagine an extreme case where the arrival rate θ_E is set to be 1, then regardless of types of product lines, their managers would like to bring forward exploratory innovation. Similar reasoning applies to an increase in the chance of drawing a high type for the new product line if implementing θ_E innovation successfully, which simply reduce the opportunity cost faced by both types of product line managers.

Corollary 1.1: *Any rewards to θ_E innovation lower n_ϕ^* :*

- *greater arrival rate of exploratory innovation ν will lower the switching index n_ϕ^*;*
- *greater probability of drawing a high type product line, h, will lower the switching index n_ϕ^*;*

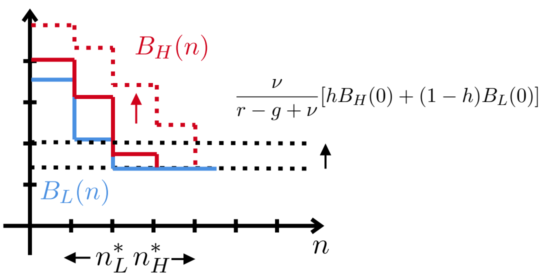
Proof: *See Appendix A.*



Comparing to the variations in ν and h, which play a role only in the 'outside' option for a product manager, an increase in the probability of entering partnership for high types has more confound implication. To be specific, a favorable change in α does increase the value of adopting θ_E innovations for both types, but it's impact on opportunity cost of two types are different. For low type product line, due to the absence of channel to partnership, a better chance to form partnership only increase the value of its outside option, which in turn reduces the opportunity cost of choosing θ_E innovation as guaranteed. However, for high type product line, at the same time, its value of maintaining the current product line also increases. It turns out that the latter force dominates.

Corollary 1.2: *A higher accessibility to H-type market, α leads to a lower switching index of low type product line, n_L^*. Furthermore, a higher α leads to a greater switching index for high type product line, n_H^*.*

Proof: *See Appendix A.*



The corollary can offer further implications for the counterfactual that we shut down the partnership, which is a limiting case where the accessibility to H-type market a decreases toward to zero. This implies the low type product line managers will conduct more incremental innovations with higher n_L^* and high type products will lower their motives in exploitative innovation with a lower n_H^* converging to n_L^*. However the impact of shutdown of partnership on the average innovation index of common input market remains ambiguous and parameter-sensitive.

3.8 Option Value of Selling Input n in Market ϕ

The last set of agents in this economy are the banker/agents. Their problem is equivalent to solving the option value of selling an input with a given innovation index n. Such value flow of the option value of selling an input is essentially composed of two parts: (1) the expected payment received from a buyer; and (2) the option value of selling the part in the next meeting. Hence the value function of $T_\phi(n, \bar{z})$ is given by:

$$rT_\phi(n, \bar{z}) = m_a^\phi \int_{x_b^\phi}^1 [P_\phi(x, n, \bar{z}) - T_\phi(n, \bar{z})]dx + \dot{T}_\phi(n, \bar{z})$$

Together with the previous lemmas, the solution to the problem reads as:

$$T_\phi = a^n t_\phi \bar{z},$$

where

$$t_L = \frac{m_a^L \omega_L A \lambda \frac{1 - {x_b^L}^2}{2}}{(r-g) + m_a^L \omega(1 - x_b^L)},$$

$$t_H = \frac{m_a^H \omega_H A \eta \frac{1 - {x_b^H}^2}{2}}{(r-g) + m_a^H \omega(1 - x_b^H)}.$$

Note that the option value is consistent to our presumption stated in previous sections. Not surprisingly, such value depends on the input market efficiency which relates to market tightness and bargaining power assignment. Furthermore, the cutoff of compatibility of input plays an important role in the evaluation of the selling option. One one hand, a greater purchase cutoff can result in a greater value-added to the firms hence a greater price to charge for sales agents. On the other hand, it may be a result of competition across sales agents, or in other words, tight input market, which may decrease the option values.

3.9 Solve the Economy at Stationary Equilibrium

After solving the problems of firms and agents, we are armed to characterize the economy as a whole. Throughout the section, we focus on solving the stationary equilibrium of the economy where the economy grows at a constant rate. We start with the determinations of the cutoffs $\{x_s^L, x_b^L, x_s^H, x_b^H\}$ which involve linking three sets of equations: (1) the indifference conditions; (2) the option value function solved previously; and (3) the market tightness at stationary equilibrium. The three sets of equations happen to coincide with all the unsolved variables in the system we presented so far. That is, at the time of reaching the solution of the cutoffs, we are finishing the last piece of the system. The indifference conditions with respect to the cutoffs are given by:

$$V_\phi(z + \eta a^n x_s^L \bar{z}, n+1, \bar{z}) = V_\phi(z + \lambda a^n x_s^L \bar{z}, n+1, \bar{z}) + T_L(n, \bar{z}),$$

The intuition of determining x_s^L is straightforward, at the cutoff, product line manager is indifferent between adopting the innovation of its own final good usage and making it for input usage. Note that both types are facing the same cutoffs.

$$V_\phi(z + \lambda a^{n_k} x_b^L \bar{z}, n, \bar{z}) - V_\phi(z, n, \bar{z}) = P_L(x, n_k, \bar{z}),$$

Similarly, to determine $\{x_b^L, x_b^H\}$ at such cutoff, it must be the case where the price of input extracts all the surplus generated for the product line.

$$V_H(z + \eta a^{n_k} x_b^H \bar{z}, n, \bar{z}) - V_H(z, n, \bar{z}) = P_H(x, n_k, \bar{z}).$$

Finally note that $x_s^H = x_s^L$ since sharing innovation with its partner is independent from its innovation usage decisions for common market.
Solving for the indifference condition, we can back out relationship between the firms' purchasing/selling decisions and option value of inputs:

$$A(\eta - \lambda)x_s^L = t_L, \; A\lambda x_b^L = t_L, \; A\eta x_b^L = t_H$$

The above equations illustrates the linear and positive relationship between cutoffs and the option value of input. A greater option value of input will render firms more incentives to make the innovation materialized for input usage, which results in a greater x_s^L. Similarly, a greater outside option of banker leads to a stronger bargaining position, which further implies a higher prices charged for the input. Hence the 'purchasing' cutoff must be higher to compensate.
To characterize the last piece: market tightness of input markets at stationary equilibrium. We solve for the invariant moving flow of input markets. For the common input market, (L), the law of motion of input follows:

$$\dot{y}_a = i x_s^L (1 - \Omega_{n_L^*}^L - \Omega_{n_H^*}^H) - m_b^L (1 - x_b^L)(1 - \Omega_{n_L^*}^L - \Omega_{n_H^*}^H)$$

where the first term on the right-hand side captures the input inflows to the common input market. Let the fraction of product lines that conduct θ_E innovation denote by $\Omega_{n_\phi^*}^\phi$. Recall that those product lines cannot access to input market. The second term reflects the parts sold to the buyers, which implies the outflow of inputs. At stationary equilibrium, we must have $\dot{n}_a = 0$, thus we have:

$$m_b^L = \frac{i x_s^L}{1 - x_b^L} = \frac{\lambda}{\eta - \lambda} \frac{i x_b^L}{1 - x_b^L},$$

which is clearly increasing in x_b^L. Furthermore, given that $m_b^L = \zeta(\frac{y_a^L}{y_b^L})^\varepsilon$, it follows that the market tightness in the common input market is given by:

$$\frac{y_a^L}{y_b^L} = (\frac{m_b^L}{\zeta_L})^{\frac{1}{\varepsilon}}$$

Hence it immediately follows that the meeting probability for sales agent is:

$$m_a^L = \zeta(\frac{m_b^L}{\zeta_L})^{\frac{\varepsilon-1}{\varepsilon}}$$

from which we can tell that m_a^L is strictly decreasing in x_b^L. Putting the three sets of equations together, we then can characterize the cutoff x_b^L:

$$x_b^L = \frac{m_a^L \omega_L \frac{1-{x_b^L}^2}{2}}{(r-g) + m_a^L \omega(1-x_b^L)},$$

which is uniquely determined.

Corollary 2.1: *A greater bargaining power of the seller ω_L leads to a higher cutoffs: $\{x_b^L, x_s^L\}$ in the common input market. Furthermore, a greater efficiency of matching process, ζ_L also increases the cutoffs.*

Proof: *See Appendix A.*

This corresponds to the two forces that push up the cutoffs. A greater bargaining power of sales agent force the buyer to wait for a better suitable inputs to compensate the higher price. On the other hand, a less frictional input market increases the meeting probability for agents who then grasp higher option value since it can trade in the next meeting with higher probability, which again increase the price of input, which further induce a higher cutoff for buyers to compensate the loss. In fact, those parameters then have deeper impact innovation strategies of firms:

Corollary 2.2: *For sufficient large bargaining power of sales agent in common market, ω_L, a greater bargaining power induces greater incentives for firms to implement θ_E innovation. Furthermore, there exists cutoff ζ^* such that for $\zeta > \zeta^*$, a more efficient input market leads to greater incentives for product lines to deviate for θ_E innovations.*

Proof: *See Appendix A.*

The result may appear to be surprising. First of all, though it is true that a more bargaining power will increase the sales income for firms who are conducing θ_D innovation, on the counterpart, it also harms the gains from sourcing for firms due to greater cost. In particular, there exists cutoffs which ensure that a higher compatibility cutoff is not enough to compensate for the greater price charged by the sales agent. Such loss lower the opportunity cost of implementing θ_E innovation due to which the product line cannot benefit from sourcing at that stage. Secondly, an increase in the sales of input for firms, though enhance the value of conducting incremental innovations, it also exacerbate the speed of decreasing return of conduct further improvement. This makes going for exploratory innovation seem more attractive since if successful, it can reset the innovation index and can benefit more from selling low index inputs.

Via same process, we can also compute for x_b^H. The final step is to compute for $\mathbb{E}_{\mu_{n_j}^{\phi}}[a^{n_j}]$. We investigate the stationary distribution of product line innovation index. Given the absence of creative destruction in the benchmark model, the stationary equilibrium the distributions are uniform for both types. Specifically, this is simply because the inflow to index n is equal to the outflow from the index n: $i\Omega_{n-1}^{\phi} = i\Omega_n^{\phi}$ for all $n \in \{1, ..., n_{\phi}^* - 1\}$. However the inflow of index $n = 0$ depends on the innovation strategies of both types. To be specific, the inflows of $n = 0$ to type H and L product line are given by:

$$\nu h(\Omega_{n_H^*}^H + \Omega_{n_L^*}^L),$$

and

$$\nu(1-h)(\Omega_{n_H^*}^H + \Omega_{n_L^*}^L),$$

while the outflow of $n = 0$ for the two types is $i\Omega_0^{\phi}$. Furthermore, the flow balance of index $n = n_H^*$ follows:

$$i\Omega_{n_{\phi}^*-1} = \nu\Omega_{n_{\phi}^*}$$

Together with fixed unit mass of product lines:

$$\sum_{k=0}^{n_H^*} \Omega_k^H + \sum_{k=0}^{n_L^*} \Omega_k^L = 1,$$

we can obtain the stationary distribution for both types:

$$\Omega_n^H = \frac{1}{n_H^* + \frac{1-h}{h} n_L^* + \frac{i}{\nu h}} \; for\ n = 0, 1, ..., n_H^* - 1,$$

$$\Omega_{n_H^*}^H = \frac{\frac{i}{v}}{n_H^* + \frac{1-h}{h} n_L^* + \frac{i}{\nu h}}$$

and

$$\Omega_n^L = \frac{\frac{1-h}{h}}{n_H^* + \frac{1-h}{h} n_L^* + \frac{i}{\nu h}} \; for\ n = 0, 1, ..., n_L^* - 1,$$

$$\Omega_{n_L^*}^L = \frac{\frac{i}{v}\frac{1-h}{h}}{n_H^* + \frac{1-h}{h} n_L^* + \frac{i}{\nu h}}$$

Given the distribution, we can compute for $\mathbb{E}_{\mu_{n_j}^{\phi}}[a^{n_j}]$:

$$\mathbb{E}_{\mu_{n_j}^L}[a^{n_j}] = \frac{1}{1 - \Omega_{n_L^*}^L - \Omega_{n_H^*}^H} \left[\sum_{n=0}^{n_H^*-1} \Omega_n^H a^n + \sum_{n=0}^{n_L^*-1} \Omega_n^L a^n \right],$$

$$\mathbb{E}_{\mu_{n_j}^H}[a^{n_j}] = \frac{1}{\sum_{n=0}^{n_H^*-1} \Omega_n^H} \sum_{n=0}^{n_H^*-1} \Omega_n^H a^n$$

Hence, together the results for the cutoffs and option values of selling parts in both markets, one is now able to compute for C_{ϕ}, D_{ϕ} and thus solve for n_L^* and n_H^*, taking the distribution of input market as given. Finally solve the fixed point problem of the distribution which in turn solves completely the flipping points. Thereafter it completes the characterization of the economy. The growth rate of the economy is summarized by the following claim:

Proposition 2: *The growth rate of the economy is given by:*

$$g = \left[\sum_{n=0}^{n_H^*-1} \Omega_n^H a^n + \sum_{n=0}^{n_L^*-1} \Omega_n^L a^n \right] \cdot \left[i(\eta \frac{1-{x_s^L}^2}{2} + \lambda \frac{{x_s^L}^2}{2}) + m_b^L \lambda \frac{1-{x_b^L}^2}{2} \right] + \sum_{n=0}^{n_H^*-1} \Omega_n^H a^n \cdot m_b^H \eta \frac{1-{x_b^H}^2}{2}$$

Proof: *See Appendix A.*

To remind, the two growth vehicles: *R&D* and sourcing from input markets contribute to the value of a given franchise product in a linear fashion in average industry productivity. Both channels are active only when the firms are conducting θ_D innovation in the benchmark set-up. By law of large number, the growth contribution from the both types via *R&D* and sourcing from common input market, L is linear in the expectation of active θ_D product line index. Similarly, the growth driven by the partnership market H is linear in its average index pool as well. As we can tell directly that the postpone in θ_E innovation will eventually slow the growth of the economy. Finally we summarize the equilibrium of this economy:

Definition 1 (Stationary Equilibrium): *A stationary equilibrium of this economy is a tuple* [14]*:*

$$\left\{ \left\{ \{x_b^\phi, x_s^\phi\}, \{n_\phi^*\}, \{m_a^\phi, m_b^\phi\}, P_\phi, T_\phi, \{\Omega_n^\phi\}_{n=0}^{n_\phi^*}, \mathbb{E}_{\mu_{n_j}^\phi}[a^{n_j}], V_\phi \right\}_{\phi \in \{L,H\}}, g, r \right\}$$

such that:

(1) $\{x_b^\phi, x_s^\phi\}_{\phi \in \{L,H\}}$, the buying and selling threshold for inputs maximizes the value of product lines ; (2) $\{n_\phi^*\}_{\phi \in \{L,H\}}$ are the optimal innovation policies solved in Lemma 2 & 3; (3) $\{m_a^\phi, m_b^\phi\}_{\phi \in \{L,H\}}$ are the input market tightness; (4) $\{P_\phi\}_{\phi \in \{L,H\}}$ are the pricing policy of input under Nash-bargaining; (5) $\{T_\phi\}_{\phi \in \{L,H\}}$ is the sales agent's problem stated in Section 4.8; (6) the stationary equilibrium distributions of incremental innovation index: $\{\{\Omega_n^\phi\}_{n=0}^{n_\phi^*}\}_{\phi \in \{L,H\}}$; (7) the average vintage of input market at the stationary equilibrium: $\{\mathbb{E}_{\mu_{n_j}^\phi}[a^{n_j}]\}_{\phi \in \{L,H\}}$; (8) the value functions of product lines $\{V_\phi\}_{\phi \in \{L,H\}}$; and (9) the growth rate stated in Proposition 2.

[14] In the Appendix, we define the equilibrium in more general manner.

Appendix (in process)

Appendix A

Derivation of Firm's Problem:

$$
e^{-rt}W(\boldsymbol{S_f}) = \left\{ \begin{array}{c}
\sum_{o\in M_L^f} \Bigg\{ e^{-rt}\Delta\pi z_{f,j_o} + \max \Bigg\{ i\Delta e^{-r(t+\Delta)} \Bigg\{ \Bigg[\int_{x_s^L}^1 W\Bigg(\boldsymbol{S_f}\backslash\{z_{j_o}, n_{j_o}\} \cup \{z_{j_o} + \eta_{n_{j_o}(x,\bar{z})}, n_{j_o}+1\}\Bigg) dx \\
+ \int_0^{x_s^L} [W\Bigg(\boldsymbol{S_f}\backslash\{z_{j_o}, n_{j_o}\} \cup \{z_{j_o} + \lambda_{n_{j_o}(x,\bar{z})}, n_{j_o}+1\}\Bigg) + T_L(n,\bar{z})]dx \Bigg\} \\
+m_b^L \Delta e^{-r(t+\Delta)} \Bigg[\mathbb{E}_{\{\mu_{n_k}^L\}} \Bigg\{ \int_{x_b^L}^1 \Bigg[[W\Bigg(\boldsymbol{S_f}\backslash\{z_{j_o}\} \cup \{z_{j_o} + \lambda_{n_j(x_{kj_o},\bar{z})}\}\Bigg)] - P(x_{kj_o}, n_k, \bar{z}) \Bigg] dx_{kj_o} \Bigg] \\
+ \int_0^{x_b^L} W(\boldsymbol{S_f})dx + (1 - i\Delta - m_b^L\Delta)e^{-r(t+\Delta)} W(\boldsymbol{S_f})_{+\Delta}; \\
\nu\Delta e^{-r(t+\Delta)} \cdot \Bigg[hW\Bigg(\boldsymbol{S_f}\backslash\{n_{j0}, H_{j_o}\} \cup \{0, H_{j_o}\}\Bigg) + (1-h)W\Bigg(\boldsymbol{S_f}\backslash\{n_{j0}, H_{j_o}\} \cup \{0, L_{j_o}\}\Bigg)\Bigg] \\
+(1-\nu\Delta)e^{-r(t+\Delta)} W(\boldsymbol{S_f})_{+\Delta} \Bigg\} \\
+\sum_{o\in M_L^f} \Bigg\{ e^{-rt}\Delta\pi z_{f,j_o} + \max \Bigg\{ i\Delta e^{-r(t+\Delta)} \Bigg\{ \Bigg[\int_{x_s^L}^1 W\Bigg(\boldsymbol{S_f}\backslash\{z_{j_o}, n_{j_o}\} \cup \{z_{j_o} + \eta_{n_{j_o}(x,\bar{z})}, n_{j_o}+1\}\Bigg) dx \\
+ \int_0^{x_s^L} [W\Bigg(\boldsymbol{S_f}\backslash\{z_{j_o}, n_{j_o}\} \cup \{z_{j_o} + \lambda_{n_{j_o}(x,\bar{z})}, n_{j_o}+1\}\Bigg) + T_L(n,\bar{z}) + \alpha\Bigg(T_H(n,\bar{z}) - F_H(n,\bar{z})\Bigg)]dx \Bigg\} \\
+m_b^L \Delta e^{-r(t+\Delta)} \Bigg[\mathbb{E}_{\{\mu_{n_k}^L\}} \Bigg\{ \int_{x_b^L}^1 \Bigg[[W\Bigg(\boldsymbol{S_f}\backslash\{z_{j_o}\} \cup \{z_{j_o} + \lambda_{n_j(x_{kj_o},\bar{z})}\}\Bigg)] - P_L(x_{kj_o}, n_k, \bar{z}) \Bigg] dx_{kj_o} \Bigg] \\
+m_b^H \Delta e^{-r(t+\Delta)} \Bigg[\mathbb{E}_{\{\mu_{n_k}^H\}} \Bigg\{ \int_{x_b^H}^1 \Bigg[[W\Bigg(\boldsymbol{S_f}\backslash\{z_{j_o}\} \cup \{z_{j_o} + \eta n_j(x_{kj_o},\bar{z})\}\Bigg)] - P_H(x_{kj_o}, n_k, \bar{z}) \Bigg] dx_{kj_o} \Bigg] \\
+ \int_0^{x_b^L} W(\boldsymbol{S_f})dx + (1 - i\Delta - m_b^L\Delta - m_b^H\Delta)e^{-r(t+\Delta)} W(\boldsymbol{S_f})_{+\Delta}; \\
\nu\Delta e^{-r(t+\Delta)} \cdot \Bigg[hW\Bigg(\boldsymbol{S_f}\backslash\{n_{j0}, H_{j_o}\} \cup \{0, H_{j_o}\}\Bigg) + (1-h)W\Bigg(\boldsymbol{S_f}\backslash\{n_{j0}, H_{j_o}\} \cup \{0, L_{j_o}\}\Bigg)\Bigg] \\
+(1-\nu\Delta)e^{-r(t+\Delta)} W(\boldsymbol{S_f})_{+\Delta} \Bigg\}
\end{array} \right\}
$$

Take $\Delta \to 0$, we obtain the HJB for the firm's problem.

Proof of Lemma 1: We prove the Lemma through *'Guess and Verify'*. Conjecture that $W(\boldsymbol{S_f}) = \sum_{m=1}^{m_f} V(z_{jm}, n_{jm}, \theta_{jm})$.
It follows:

$$
\begin{aligned}
(A.1) \qquad & W\Bigg(\boldsymbol{S_f}\backslash\{z_{jm}, n_{jm}\} \cup \{z_{jm} + \eta_{n_{jm}(x,\bar{z})}, n_{jm}+1\}\Bigg) - W(\boldsymbol{S_f}) \\
& = V(z_{jm} + \eta_{n_{jm}(x,\bar{z})}, n_{jm}+1, \bar{z}) - V(z_{jm}, n_{jm}\bar{z});
\end{aligned}
$$

$$
\begin{aligned}
(A.2) \qquad & W\Bigg(\boldsymbol{S_f}\backslash\{z_{jm}, n_{jm}\} \cup \{z_{jm} + \lambda_{n_{jm}(x,\bar{z})}, n_{jm}+1\}\Bigg) - W(\boldsymbol{S_f}) + T(n_{jm}\bar{z}) \\
& = V(z_{jm} + \lambda_{n_{jm}(x,\bar{z})}, n_{jm}+1, \bar{z}) - V(z_{jm}, n_{jm}, \bar{z}) + T(n_{jm}\bar{z});
\end{aligned}
$$

$$
(A.3) \quad \begin{aligned} &\mathbb{E}_{\{\mu^{\phi}_{n_k}\}}\left[W\left(\boldsymbol{S_f}\backslash\{z_{jm}\}\cup\{z_{jm}+S^{\phi}_{n_k}(x_{kjm},\bar{z})\}\right)-P_{\phi}(x_{kjm},n_k,\bar{z})-W(\boldsymbol{S_f})\right] \\ &=\mathbb{E}_{\{\mu^{\phi}_{n_k}\}}\left[[V(z_{jm}+S^{\phi}_{n_k}(x_{kjm},\bar{z}),n_k,\bar{z})]-V(z_{jm},n_{jm},\theta_{jm})-P(x_g,z_{jm},\bar{z})\right]; \end{aligned}
$$

$$
(A.4) \quad \begin{aligned} &hW\left(\boldsymbol{S_f}\backslash\{n_{jm},\phi_{jm}\}\cup\{0,H_{jm}\}\right)+(1-h)W\left(\boldsymbol{S_f}\backslash\{n_{jm},\phi_{jm}\}\cup\{0,L_{jm}\}\right)-W(\boldsymbol{S_f}) \\ &=hV_H(z_{jm},0,\bar{z})+(1-h)V_L(z_{jm},0,\bar{z})-V_{\phi_{jm}}(z_{jm},n_{j,m},\bar{z}) \end{aligned}
$$

above which the equations show that the firms' value flow can be summarized at product line level. Therefore we can further rewrite firm's problem by substituting with (A.1-4).

$$
r\sum_{m=1}^{m_f}V_{\phi_{jm}}(z_{jm},n_{jm},\bar{z})-\sum_{m=1}^{m_f}\dot{V}_{\phi_{jm}}(z_{jm},n_{jm},\bar{z})=\left\{\begin{array}{c}
\sum_{m\in M^f_L}\Big[\pi z_{jm}+\max_{\{\theta_D,\theta_E\}}\Big\{i\Big\{\int_{x^L_s}^1[V_L(z_{jm}+\eta a^n_{jm}x\bar{z},n_{jm}+1,\bar{z})-V_L(z_{jm},n_{jm},\bar{z})]dx \\
+\int_0^{x^L_s}[V_L(z_{jm}+\lambda a^n_{jm}x\bar{z},n_{jm}+1,\bar{z})-V_L(z_{jm},n_{jm},\bar{z})+T_L(n_{jm},\bar{z})]dx\Big\} \\
+m^L_b\mathbb{E}_{\{\mu^L_{n_k}\}}\Big\{\int_{x^L_b}^1[V_L(z_{jm}+\lambda a^{n_k}x_{kjm}\bar{z},n_{jm},\bar{z})-V_L(z_{jm},n_{jm},\bar{z})-P_L(x_{kjm},n_k,\bar{z})]dx_{kjm}\Big\}; \\
\nu\Big[hV_H(z_{jm},0,\bar{z})+(1-h)V_L(z_{jm},0,\bar{z})-V_L(z_{jm},n_{jm},\bar{z})\Big]\Big\}\Big] \\
+\sum_{m\in M^f_H}\Big[\pi z_{jm}+\max_{\{\theta_D,\theta_E\}}\Big\{i\Big\{\int_{x^H_s}^1[V_H(z_{jm}+\eta a^n_{jm}x\bar{z},n_{jm}+1,\bar{z})-V_H(z_{jm},n_{jm},\bar{z})]dx \\
+\int_0^{x^H_s}[V_L(z_{jm}+\lambda a^n_{jm}x\bar{z},n_{jm}+1,\bar{z})-V_H(z_{jm},n_{jm},\bar{z})+T_L(n_{jm},\bar{z})]dx+\alpha\Big(T_H(n_{jm},\bar{z})-F_H(n_{jm},\bar{z})\Big)\Big\} \\
+m^L_b\mathbb{E}_{\{\mu^L_{n_k}\}}\Big\{\int_{x^L_b}^1[V_L(z_{jm}+\lambda a^{n_k}x_{kjm}\bar{z},n_{jm},\bar{z})-V_L(z_{jm},n_{jm},\bar{z})-P_L(x_{kjm},n_k,\bar{z})]dx_{kjm}\Big\} \\
+m^H_b\mathbb{E}_{\{\mu^H_{n_k}\}}\Big\{\int_{x^H_b}^1[V_H(z_{jm}+\eta a^{n_k}x_{kjm}\bar{z},n_{jm},\bar{z})-V_H(z_{jm},n_{jm},\bar{z})-P_H(x_{kjm},n_k,\bar{z})]dx_{kjm}\Big\}; \\
\nu\Big[hV_H(z_{jm},0,\bar{z})+(1-h)V_L(z_{jm},0,\bar{z})-V_H(z_{jm},n_{jm},\bar{z})\Big]\Big\}
\end{array}\right\}
$$

which clearly verifies the conjecture. □

Proof of Lemma 2: Combine with the presumption upon $T(n,\bar{z})$, we conjecture that $V_L(z,n,\bar{z})=Az+B(n)\bar{z}$. Firstly, we split the problem into two hypocritical scenarios:

$$
A(1).\ rV_L(z,n,\bar{z})-\dot{V}_L(z,n,\bar{z})=\left\{\begin{array}{c}
\pi z+\Big\{i\Big\{\int_{x^L_s}^1\big[V_L(z+\eta a^n x\bar{z},n+1,\bar{z})-V_L(z,n,\bar{z})\big]dx \\
+\int_0^{x^L_s}\big[V_L(z+\lambda a^n x\bar{z},n+1,\bar{z})-V_L(z,n,\bar{z})+T_L(n,\bar{z})\big]dx\Big\} \\
+m^L_b\mathbb{E}_{\{\mu^L_{n_j}\}}\Big\{\int_{x^L_b}^1\big[V_L(z+\lambda a^{n_j}x\bar{z},n,\bar{z})-V_L(z,n,\bar{z})-P_L(x,n_j,\bar{z})\big]dx\Big\}
\end{array}\right\}
$$

$$
A(2).\ rV_L(z,n,\bar{z})-\dot{V}_L(z,n,\bar{z})=\left\{\pi z+\nu\Big[hV_H(z,0,\bar{z})+(1-h)V_L(z,0,\bar{z})-V_L(z,n,\bar{z})\Big]\right\}
$$

Lemma A1: Let the solutions to the two problems denote by $V^1_L(z,n,\bar{z})$ and $V^2_L(z,n,\bar{z})$, given the presumption: $T_L(n,\bar{z})=t_La^n\bar{z}$, we have:

$$
V^1_L(z,n,\bar{z})=Az+\hat{B}_L(n)\bar{z},
$$

$$
V^2_L(z,n,\bar{z})=Az+\tilde{B}_L(n)\bar{z},
$$

Proof: Conjecture that $V^1_L(z,n,\bar{z})=Az+\hat{B}_L(n)\bar{z}$, it follows, the left-hand-side of Problem

$A(1)$ reads as:

$$rV_L(z,n,\bar{z}) - \dot{V}_L(z,n,\bar{z}) = rAz + (r-g)\hat{B}_L(n)\bar{z}$$

The term within the bracket before θ_D innovation arrival i is then given by:

$$\int_{x_s^L}^{1} [A\eta a^n x + \hat{B}_L(n+1) - \hat{B}_L(n)]\bar{z}dx + \int_{0}^{x_s^L} [A\lambda a^n x + \hat{B}_L(n+1) - \hat{B}_L(n)]\bar{z}dx$$
$$= \left[a^n C_L + \hat{B}_L(n+1) - \hat{B}_L(n)]\right]\bar{z}$$

Recall that the price of an input with index n is a weighted average of its value-added to the buyer and the option value of selling the input in the next meeting:

$$P_L(x,n_j,\bar{z}) = \omega\left[V_L(z+\lambda a^{n_j}x\bar{z},n,\bar{z}) - V_L(z,n,\bar{z})\right] + (1-\omega)T_L(n_j,\bar{z}),$$

This follows:

$$m_b^L \mathbb{E}_{\{\mu_{n_j}^L\}}\left\{\int_{x_b^L}^{1} \left[V_L(z+\lambda a^{n_j}x\bar{z},n,\bar{z}) - V_L(z,n,\bar{z}) - P_L(x,n_j,\bar{z})\right]dx\right\}$$
$$= (1-\omega)m_b^L \mathbb{E}_{\{\mu_{n_j}^L\}}\left\{\int_{x_b^L}^{1} [A\lambda a^{n_j}x - t_L a^{n_j}]\bar{z}dx\right\} = m_b^L D_L \bar{z}.$$

Hence the right-hand side of $A(1)$ equals to:

$$\pi z + i\left[a^n C_L + \hat{B}_L(n+1) - \hat{B}_L(n)\right]\bar{z} + m_b^L D_L \bar{z},$$

Equating both sides term by term, we obtain:

$$A = \frac{\pi}{r},$$

$$(r-g)\hat{B}_L(n) = i\left[a^n C_L + \hat{B}_L(n+1) - \hat{B}_L(n)\right] + m_b^L D_L.$$

Similarly for problem $A(2)$, according to our guess, left-hand side is then given by:

$$rAz + (r-g)\tilde{B}_L(n)\bar{z},$$

Right-hand side of the problem is expanded as:

$$\pi z + \nu\left[hB_H(0) + (1-h)B_L(0) - \tilde{B}_L(n)\right]\bar{z}.$$

Equating both sides term by term:

$$A = \frac{\pi}{r},$$

$$(r-g)\tilde{B}(n) = \nu\left[hB_H(0) + (1-h)B_L(0) - \tilde{B}_L(n)\right],$$

above which completes the claim. $\square$

Since either innovation strategies will lead to the same solution format, it implies that for the solution to problem (2), we must have:

$$V_L(z, n, \bar{z}) = Az + B_L(n)\bar{z}$$

Now we further move to characterize the solution $B_L(n)$ through the next lemma.

Lemma A2: There exists a flipping point n_L^* such that:

$$(r-g)B_L(n) = \begin{cases} i[a^n C_L + B_L(n+1) - B_L(n)] + m_b^L D_L & \textit{if } n < n_L^* \\ \nu[hB_H(0) + (1-h)B_L(0) - B_L(n)] & \textit{if } n \geq n_L^* \end{cases}$$

Proof: By contradiction, suppose instead that for the product line manager, it is optimal to choose θ_D incremental innovation forever. Then the problem is simply written as Problem $A(1)$. We claim that $A(1)$ constitute a contraction mapping by verifying it satisfies Blackwell's Condition. Rearrange the equation associated with $\hat{B}_L(n)$ shown previously, we define a mapping $\mathbb{T}$ such that

$$\mathbb{T}[\hat{B}_L(n)] = \frac{i}{\rho+i}[a^n C_L + \hat{B}_L(n+1)] + \frac{1}{\rho+i} m_b^L D_L$$

where $\rho = r - g$ is the discount rate under logarithmic preference. It is easy to verify that the monotonicity condition satisfies: Let $G(n) > B(n)$ for all n, then we have $\mathbb{T}[G(n)] > \mathbb{T}[B(n)]$. To check discounting condition, observe that:

$$\mathbb{T}[\hat{B}_L(n)+c] = \frac{i}{\rho+i}[a^n C_L + \hat{B}_L(n+1)] + \frac{i}{\rho+i}c + \frac{1}{\rho+i} m_b^L D_L < \mathbb{T}[\hat{B}_L(n)] + c, \textit{ for all } c > 0.$$

Hence $\mathbb{T}$ is a well-defined contraction mapping, which implies $[B(n) - B(n+1)]$ diminishes as $n \to \infty$. In particular,

$$\hat{B}_L(\infty) = \frac{i}{\rho+i}\hat{B}_L(\infty) + \frac{1}{\rho+i} m_b^L D_L(\infty) \Rightarrow \hat{B}_L(\infty) = \frac{m_b^L D_L(\infty)}{\rho} = 0$$

The last equality comes from the fact that within D_L, $\mathbb{E}_{\{\mu_{n_j}^L\}}[a^{n_j}] = 0$ as $n \to 0$ in stationary equilibrium, which we have shown in Section 3.9. While on the other hand, the solution to Problem $A(2)$ sugguests that:

$$\rho\tilde{B}_L(n) = \nu[hB_H(0) + (1-h)B_L(0) - \tilde{B}_L(n)]$$

of which $\tilde{B}_L(n)$ is strictly positive and constant. This implies there exists a cutoff n^* such that for all $n \geq n^*$. $i[a^n C_L + \hat{B}_L(n+1) - \hat{B}_L(n)] + m_b^L D_L \leq \nu[hB_H(0) + (1-h)B_L(0) - \tilde{B}_L(n)]$. To complete proof, define $b(n) \equiv a^n C_L + B_L(n+1) - \frac{i-\nu}{i} B_L(n)$, this follows:

$$(\rho+\nu)B_L(n) = \max\left\{ ib(n) + m_b^L D_L, \nu[hB_H(0) + (1-h)B_L(0)] \right\},$$

$$(\rho+\nu)\frac{i}{i-\nu}B_L(n+1) = \frac{i}{i-\nu}\max\left\{ib(n+1) + m_b^L D_L, \nu[hB_H(0) + (1-h)B_L(0)]\right\},$$

Take difference of the above two equations, we have:

$$\frac{i}{i-\nu}(\rho+\nu)b(n) + \max\left\{ib(n) + m_b^L D_L; \nu[hB_H(0) + (1-h)B_L(0)]\right\}$$
$$= \frac{i}{i-\nu}\left[(\rho+\nu)a^n C_L + \max\left\{ib(n+1) + m_b^L D_L; \nu[hB_H(0) + (1-h)B_L(0)]\right\}\right]$$

Let $F(b(n)) \equiv \frac{i}{i-\nu}(\rho+\nu)b(n) + \max\left\{ib(n) + m_b^L D_L; \nu[hB_H(0) + (1-h)B_L(0)]\right\}$, and it is easy to see $F(\cdot)$ is strictly increasing and piece-wise linear. Hence its inverse is also strictly increasing and piece-wise linear. Hence we define the mapping T such that:

$$T[b(n)] = F^{-1}[\frac{i}{i-\nu}\left[(\rho+\nu)a^n C_L + \max\left\{ib(n+1) + m_b^L D_L; \nu[hB_H(0) + (1-h)B_L(0)]\right\}\right]];$$

which clearly satisfies monotonicity condition. We claim the discounting condition holds as well. Suppose that $i(b(n+1)+c) + m_b^L D_L \leq \nu[hB_H(0) + (1-h)B_L(0)]$, it immediately follows:

$$T[b(n)+c] = F^{-1}[\frac{i}{i-\nu}\left[(\rho+\nu)a^n C_L + \nu[hB_H(0) + (1-h)B_L(0)]\right]]$$
$$= T[b(n)]$$

Otherwise, we have:

$$T[b(n)+c] = F^{-1}[\frac{i}{i-\nu}\left[(\rho+\nu)a^n C_L + ib(n+1) + m_b^L D_L + ic\right]$$
$$< T[b(n)] + \frac{\frac{i}{i-\nu}ic}{\frac{i}{i-\nu}(\rho+\nu)+i} = T[b(n)] + \frac{i}{\rho+i}c < T[b(n)] + c$$

Therefore, T is a contraction mapping and we have obtained the desired results. □

Proof of Lemma 3: The proof follows the same fashion as we have shown in that of Lemma 2. The only difference is that we have an extra term attached to value function if implementing θ_D innovation. We skip the computations here. □

Proof of Proposition 1:

$$A(3).(r-g)B_L(n) = \begin{cases} i[a^n C_L + B_L(n+1) - B_L(n)] + m_b^L D_L & \textit{if } n < n_L^* \\ \nu[hB_H(0) + (1-h)B_L(0) - B_L(n)] & \textit{if } n \geq n_L^* \end{cases}$$

$$A(4).(r-g)B_H(n) = \begin{cases} i[a^n C_L + B_H(n+1) - B_H(n)] + m_b^L D_L + m_b^H D_H & \textit{if } n < n_H^* \\ \nu[hB_H(0) + (1-h)B_L(0) - B_H(n)] & \textit{if } n \geq n_H^* \end{cases}$$

We characterize the above two problem in an uniform way. Similar to what we have defined before, let $L(n) \equiv a^n C_L + B(n+1) - \frac{i-\nu}{i} B(n)$, and let $H(n) \equiv a^n C_L + B(n+1) - \frac{i-\nu}{i} B(n) + \frac{m_b^H D_H}{i}$. The above two solutions correspond to the mapping:

$$T[L(n)] = F^{-1}[\frac{i}{i-\nu}\Bigg[(\rho+\nu)a^n C_L + \max\Bigg\{iL(n+1) + m_b^L D_L; \nu[hB_H(0) + (1-h)B_L(0)]\Bigg\}\Bigg]],$$

$$T[H(n)] = F^{-1}[\frac{i}{i-\nu}\Bigg[(\rho+\nu)(a^n C_L + \frac{m_b^H D_H}{i}) + \max\Bigg\{iH(n+1) + m_b^L D_L; \nu[hB_H(0) + (1-h)B_L(0)]\Bigg\}\Bigg]],$$

where F^{-1} is identical to what we have shown in the previous lemma. Given monotonicity of F^{-1} it is easy to conclude that $T(H(n)) \geq T(L(n))$ for all n, which then, together with $A(3)$ and $A(4)$, it follows that:

$$(\rho+\nu)B_H(n) \geq (\rho+\nu)B_L(n) \Rightarrow B_H(n) \geq B_L(n) \text{ for any } m_b^H D_H > 0$$

Now suppose that $n_L^* > n_H^*$. By definition of flipping point n_L^*, n_H^*, we have

$$B_H(n_H^*) = B_L(n_L^*)$$

Since $B_L(\cdot)$ is strictly decreasing before n_L^*, it follows: $B_L(n_L^*) < B_L(n_H^*)$, which implies that $B_L(n_H^*) > B_H(n_H^*)$, leading to a contradiction. Hence we conclude that $n_H^* \geq n_L^*$. □

Proof of Corollary 1: We analyze the comparative statics over $\{\nu, h, a\}$, which is independent from the terms influenced by general equilibrium channel. We firstly investigate the impact of arrival rate of θ_E innovation, ν on the innovation strategies of both types of product lines. Given the solutions derived from Lemma 2 and Lemma 3, we can rewrite $B_L(0), B_H(0)$ by forward iterating:

$$\begin{aligned} B_L(0) &= \frac{C_L}{a}\sum_{k=1}^{n_L^*}(\frac{ia}{\rho+i})^k + (\frac{i}{\rho+i})^{n_L^*} B_L(n_L^*) + \frac{m_b^L D_L}{i}\sum_{k=1}^{n_L^*}(\frac{i}{\rho+i})^k \\ &\equiv L_C + (\frac{i}{\rho+i})^{n_L^*} B_L(n_L^*) + L_D, \end{aligned}$$

$$\begin{aligned} B_H(0) &= \frac{C_L}{a}\sum_{k=1}^{n_H^*}(\frac{ia}{\rho+i})^k + (\frac{i}{\rho+i})^{n_H^*} B_H(n_H^*) + \frac{m_b^L D_L + m_b^H D_H}{i}\sum_{k=1}^{n_H^*}(\frac{i}{\rho+i})^k \\ &\equiv H_C + (\frac{i}{\rho+i})^{n_H^*} B_H(n_H^*) + H_D \end{aligned}$$

Note that $B_L(n_L^*) = B_H(n_H^*) = \frac{\rho}{\rho+\nu}[hB_H(0) + (1-h)B_L(0)] \equiv \frac{\rho}{\rho+\nu}\tilde{B}(0)$.
This follows:

$$\tilde{B}(0) = \frac{(1-h)(L_C + L_D) + h(H_C + H_D)}{1 - \frac{\nu}{\rho+\nu}\Bigg[(1-h)(\frac{i}{\rho+i})^{n_L^*} + h(\frac{i}{\rho+i})^{n_H^*}\Bigg]}$$

It is easy to see that, for any fixed pair (n_L^*, n_H^*), $\tilde{B}(0)$ is increasing in ν. Now consider the case where $n = n_L^* - 1$, hence it is optimal it conduct θ_D innovation:

$$i[a^{n_L^*-1}C_L + B_L(n_L) - B_L(n_L^* - 1)] + m_b^L D_L > \nu(\tilde{B}(0) - B_L(n_L^* - 1))$$

Let $F_L \equiv i[a^{n_L^*-1}C_L + B_L(n_L) - B_L(n_L^*-1)] + m_b^L D_L - \nu(\tilde{B}(0) - B_L(n_L^*-1))$ capturing the gap between the value-added from θ_D innovation and θ_E innovation for low type product line. Together with $B_L(n_L^*) = \frac{\nu}{\rho+\nu}\tilde{B}(0)$, we have:

$$F_L = \frac{i}{\rho+i}a^{n_L^*-1}C_L - \frac{\rho}{\rho+i}\frac{\nu}{\nu+\rho}\tilde{B}(0) + \frac{1}{\rho+i}m_b^L D_L$$

Clearly F is strictly decreasing in ν, which implies that for sufficiently large ν, the gap will be negative, which implies a decrease in n_L^*. Same result holds for n_H^*.
Secondly, we investigate the impact of variation in h on firms' innovation decision. Notice that $B_H(0) > B_L(0)$ as we have shown in Proposition 1. This implies $\frac{\partial \tilde{B}(0)}{\partial h} > 0$, which immediately follows that F_L is decreasing in h. Hence an increase in h induce stronger incentives for low type product line to implement θ_E innovation strategy. Again, same result holds for high type product line.
Third, we investigate the influence of changing α, the rate of accessing to partnership upon the innovation direction. Note that $\frac{\partial B_L(0)}{\partial \alpha} = 0$ as low type cannot access to partnership at all, while $\frac{\partial B_H(0)}{\partial \alpha} > 0$. This implies $\frac{\partial \tilde{B}(0)}{\partial \alpha} > 0$, and therefore a higher accessibility to partnership lower n_L^*. However its impact on the innovation decision of high type product line remains ambiguous. Intuitively, a greater α increases the incentives of implementing θ_D innovation as it brings more growth through partnership. On the other hand, it also increases the option value of directing to θ_E innovation. Consider:

$$F_H = \frac{i}{\rho+i}a^{n_L^*-1}C_L - \frac{\rho}{\rho+i}\frac{\nu}{\nu+\rho}\tilde{B}(0) + \frac{1}{\rho+i}m_b^L D_L + \frac{1}{\rho+i}m_b^H D_H$$

$$\frac{\partial F_H}{\partial \alpha} = \left\{\frac{1}{\rho+i} - \frac{\rho}{\rho+i}\frac{\nu}{\nu+\rho}\frac{h}{i}\frac{\sum_{k=1}^{n_H^*}(\frac{i}{\rho+i})^k}{1 - \frac{\nu}{\rho+\nu}\left[(1-h)(\frac{i}{\rho+i})^{n_L^*} + h(\frac{i}{\rho+i})^{n_H^*}\right]}\right\}\frac{\partial m_b^H D_h}{\partial \alpha}$$

$$= \frac{1}{\rho+i}\frac{\partial m_b^H D_h}{\partial \alpha}\{1 - \frac{\frac{\nu}{\rho+\nu}h[1-(\frac{i}{\rho+i})^{n_H^*}]}{1 - \frac{\nu}{\rho+\nu}\left[(1-h)(\frac{i}{\rho+i})^{n_L^*} + h(\frac{i}{\rho+i})^{n_H^*}\right]}\}$$

Note that $\frac{\frac{\nu}{\rho+\nu}h[1-(\frac{i}{\rho+i})^{n_H^*}]}{1-\frac{\nu}{\rho+\nu}\left[(1-h)(\frac{i}{\rho+i})^{n_L^*}+h(\frac{i}{\rho+i})^{n_H^*}\right]} < h\frac{1-(\frac{i}{\rho+i})^{n_H^*}}{1-\left[(1-h)(\frac{i}{\rho+i})^{n_L^*}+h(\frac{i}{\rho+i})^{n_H^*}\right]}$, from which we can infer that there exists a h^* for all $h < h^*$ we have:

$$h\frac{1-(\frac{i}{\rho+i})^{n_H^*}}{1-\left[(1-h)(\frac{i}{\rho+i})^{n_L^*} + h(\frac{i}{\rho+i})^{n_H^*}\right]} < 1$$

In such cases, we have $\frac{\partial F_H}{\partial \alpha} > 0$. However, the result is still too restricted. We pick up the results from contraction mapping to show a stronger prediction:
Revisit the contraction mapping T defined in **Proposition 1**, and let $\tilde{H}(n) \equiv a^n C_L + B(n+1) - \frac{i-\nu}{i} B(n) + \frac{m_b^H \hat{D}_H}{i}$, where $\hat{D}_H(\hat{\alpha}) > D_H(\alpha)$ with $\hat{\alpha} > \alpha$. Hence it follows that $B_H(n;\hat{\alpha}) \geq B_H(n;\alpha)$. With the same technique applied in Proposition 1, we can conclude n_H^* weakly increases. □

Proof of Corollary 2.1: Link the option value of an input and the cutoff indifference decision together, we have:

$$x_b = \frac{m_a^L \omega_L \frac{1-x_b^{L\,2}}{2}}{\rho + m_a^L \omega (1 - x_b^L)},$$

After rearrangement,

$$\frac{\rho + m_a^L \omega}{m_a^L \omega} = \frac{1 + x_b^{L\,2}}{2 x_b^L}$$

Since m_b^L is increasing in x_b^L, it follows m_a^L is decreasing in x_b^L. Hence the left-hand side of equation is increasing in x_b^L. On the right-hand side, since $x_b^L \in [0,1]$, it is decreasing in x_b^L. Hence an increase in ω leads to a shift down of LHS, which causes an increase in x_b^L. Similarly, an increase in ζ will push up m_a^L, and thus shifts down the LHS, which again leads to an increase in x_b^L. □

Proof of Corollary 2.2: First we take the first-order approach to investigate the variation of C_L, D_L by treating them individually. To be specific, given the solution form of product line problem, we can alternatively re-interpret the optimal innovation choice is the one that maximizes the initial value stream $B_\phi(0)$ in principal. We relax the integer constraints upon the flipping point, and enforce the first order condition with respect to n_ϕ^* to take zero, and explore the impact of changing C_L and D_L on the first order condition then. To simplify the outlook, we abuse the notation by letting $n_\phi^* \equiv \phi$, $C_L \equiv C$, and $m_b^L D_L \equiv D$, $m_b^H D_H \equiv d$. Firstly we investigate $\frac{\partial \tilde{B}(0)}{\partial L}$: The nominator of the derivative reads as:

$$(1-h)[\frac{ci}{\rho+i-ia}[log\frac{\rho+i}{ia}(\frac{ai}{\rho+i})^L] + \frac{D}{\rho}log\frac{\rho+i}{i}(\frac{i}{\rho+i})^L][1 - \frac{\nu}{\rho+\nu}[(1-h)(\frac{i}{\rho+i})^L + h(\frac{i}{\rho+i})^H]]$$
$$- \frac{\nu}{\rho+\nu}(1-h)[log\frac{\rho+i}{i}(\frac{i}{\rho+i})^L][(1-h)(L_C + L_D) + h(H_C + H_D)]$$

We collect the terms associated with D and obtain:

$$(1-h)\Big\{\frac{D}{\rho}log\frac{\rho+i}{i}(\frac{i}{\rho+i})^L + \frac{D}{\rho}\frac{\nu}{\rho+\nu}log\frac{\rho+i}{i}(\frac{i}{\rho+i})^L[(1-h)(\frac{i}{\rho+i})^L + h(\frac{i}{\rho+i})^H]$$
$$- \frac{\nu}{\rho+\nu}\frac{D}{\rho}log\frac{\rho+i}{i}(\frac{i}{\rho+i})^L - \frac{D}{\rho}\frac{\nu}{\rho+\nu}log\frac{\rho+i}{i}(\frac{i}{\rho+i})^L[(1-h)(\frac{i}{\rho+i})^L + h(\frac{i}{\rho+i})^H]\Big\}$$
$$= (1-h)\frac{\rho}{\rho+\nu}\frac{D}{\rho}log\frac{\rho+i}{i}(\frac{i}{\rho+i})^L > 0$$

This immediately follows that at 'equilibrium' $\frac{\partial B_L(0)}{\partial L} = 0$, an increase in $m_b^L D_L$ leads to a push up to n_L^*. Similarly we can conclude that an increase in $m_b^H D_H$ leads to a push down on n_L^*. Recall that n is restricted to be integer, this implies at the first order condition at n_L^* must be non-positive. It follows that an increase in C_L has a negative impact. Similar results hold for the high type except for that an increase in $m_b^H D_H$ induces more incentives to keep on the track of θ_D innovations as $\frac{\partial B_H(0)}{\partial H}$ is increasing in $m_b^H D_H$.
Given the fact that the flipping point is decreasing in C_L and increasing in $m_b^L D_L$, we then investigate how change in bargaining power parameter ω_L influence both terms. Recall that $C_L = A\left(\eta \frac{1-{x_s^L}^2}{2} + \lambda \frac{{x_s^L}^2}{2}\right) + x_s^L t_L$, substitute out t_L, one can obtain:

$$C_L = \frac{A}{2}[\eta + \lambda + (\eta - \lambda){x_s^L}^2]$$

which is clearly increasing in x_b^L, hence is driven up by an increase in ω_L and ζ_L. Expand D_L:

$$D_L = (1-\omega_L)\mathbb{E}_{\{\mu_{n_j}^L\}}[a^{n_j}]\left[A\lambda\frac{1-{x_b^L}^2}{2} - (1-x_b^L)t_L\right] = (1-\omega_L)\mathbb{E}_{\{\mu_{n_j}^L\}}[a^{n_j}]\frac{A\lambda}{2}(1-x_b^L)^2,$$

which is decreasing when $x_b^L \geq \frac{1}{2}$. Hence for sufficiently large ω_L or ζ_L, it will derive $x_b^L > \frac{1}{2}$ and induce an increase in D_L. Overall they will lead to increase in C_L and decrease in D_L which cause a greater incentives to θ_E innovations. □

Proof of Proposition 2: As we have proceed in the section 3.9, we have shown the equilibrium exists. To see the uniqueness of the equilibrium, it is sufficient to show that there exists a pair (n_L^*, n_H^*) solve the product line problems of two types. Take the input index distribution as exogenously given, by finite dimensional contraction mapping, we are able to apply Kakutani fixed point theorem, which implies the uniqueness of the solution to the mapping. To illustrate the growth of the economy, consider the following law of motion:

$$\begin{aligned}\bar{z}_{+\Delta} = \bar{z} + \Delta \int_{j\in\{j:\phi_j=L, n_j<n_L^*\}} &\left[i[\int_{x_s^L}^1 a^{n_j}\eta x\bar{z}dx + \int_0^{x_s^L} a^{n_j}\lambda x\bar{z}dx] + m_b^L \mathbb{E}_{\mu_{n_k}^L}\int_{x_b^L}^1 \lambda a^{n_k}x\bar{z}dx\right]dj \\ + \Delta \int_{j\in\{j:\phi_j=H, n_j<n_H^*\}} &\left[i[\int_{x_s^L}^1 a^{n_j}\eta x\bar{z}dx + \int_0^{x_s^L} a^{n_j}\lambda x\bar{z}dx] + m_b^L \mathbb{E}_{\mu_{n_k}^L}\int_{x_b^L}^1 \lambda a^{n_k}x\bar{z}dx \right.\\ &\left. + m_b^H \mathbb{E}_{\mu_{n_k}^H}[a^{n_k}]\int_{x_b^H}^1 \eta a^{n_k}x\bar{z}dx\right]dj\end{aligned}$$

which can be further written as:

$$\begin{aligned}\bar{z}_{+\Delta} = \bar{z} + [1 - \Omega_{n_L^*}^L - \Omega_{n_H^*}^H]\Delta\mathbb{E}_{\mu_{n_k}^L}[a^{n_k}]\cdot\left[i(\eta\frac{1-{x_s^L}^2}{2} + \lambda\frac{{x_s^L}^2}{2}) + m_b^L\lambda\frac{1-{x_b^L}^2}{2}\right]\bar{z} \\ + \sum_{n=0}^{n_H^*-1}\Omega_n^H\Delta\mathbb{E}_{\mu_{n_j}^H}[a^{n_j}]m_b^H\eta\frac{1-{x_b^H}^2}{2}\bar{z}\end{aligned}$$

Take $\Delta \to 0$, we obtain the growth rate:

$$g = [1 - \Omega^L_{n^*_L} - \Omega^H_{n^*_H}]\mathbb{E}_{\mu^L_{n_k}}[a^{n_k}] \cdot \left[i(\eta \frac{1 - {x^L_s}^2}{2} + \lambda \frac{{x^L_s}^2}{2}) + m^L_b \lambda \frac{1 - {x^L_b}^2}{2} \right] \bar{z}$$
$$+ \sum_{n=0}^{n^*_H - 1} \Omega^H_n \mathbb{E}_{\mu^H_{n_j}}[a^{n_j}] m^H_b \eta \frac{1 - {x^H_b}^2}{2} \bar{z}$$

□

More discussions on the Notion of Equilibrium: We specify the definition of equilibrium in response to the restriction on the integer space of innovation index n. As mentioned before, the innovation policies are made by taking the average vintage of input market as given. This implies the existence of equilibrium must ensure the existence of fixed point that maps $\{n^*_L, n^*_H\}$ into $\mathbb{E}_{\mu^\phi_{n_j}}[a^{n_j}]$. To do this, we allow some of firms choose to switch to θ_E innovation at n^*_ϕ while some switch to θ_E at $n^*_\phi + 1$ in the spirit of perfect foresight equilibrium by Jovanovic and Lach (1989). This implies at n^*_ϕ, the product line managers with type ϕ are indifferent between switching or not. Let q_ϕ be the fraction of ϕ type product lines that switch at n^*_ϕ, and $1 - q_\phi$ is the fraction of those switch at $n^*_\phi + 1$. Given the construction, we permit continuous structure over $\mathbb{E}_{\mu^\phi_{n_j}}[a^{n_j}]$ between n^*_ϕ and $n^*_\phi + 1$. To see this, consider the flow balance in low type product lines at n^*_L at a stationary distribution:

$$i\Omega^L_{n^*_L - 1} = vq_L \Omega^L_{n^*_L} + i(1 - q_L)\Omega^L_{n^*_L}$$

Note that the outflow of n^*_L input is composed of two parts: q_L of $\Omega^L_{n^*_L}$ switches to $n = 0$ by implementing θ_E innovation; $1 - q_L$ of $\Omega^L_{n^*_L}$ keeps on θ_D innovation as product lines are indifferent between switching to θ_E and keeping θ_D at equilibrium. The flow balance of $n^*_L + 1$ is then given by:

$$i(1 - q_L)\Omega^L_{n^*_L} = \nu \Omega^L_{n^*_L + 1}$$

Similar to what we have shown previously, the flow balance of $n = 0$ is given by:

$$\nu(1 - h)\left[q_L \Omega^L_{n^*_L} + \Omega^L_{n^*_L + 1} + q_H \Omega^L_{n^*_H} + \Omega^H_{n^*_H + 1} \right] = i\Omega^L_0$$

The stationary distribution is then characterized by:

$$\Omega^H_n = \frac{1}{n^*_H + \frac{1-h}{h} n^*_L + \frac{i}{v}\left[\frac{v + i(1-q_H)}{vq_H + i(1-q_H)} + \frac{v + i(1-q_L)}{vq_L + i(1-q_L)} \frac{1-h}{h}\right]} \quad \textit{for } n = 0, 1, ..., n^*_H - 1,$$

Note that if $q_L = q_H = 1$, we have the same result shown in Section 4.9. Similarly, if $q_L = q_H = 0$, then the above distribution is equivalence to the case where the swichting points are $\{n^*_H + 1, n^*_L + 1\}$.

$$\Omega^H_{n^*_H} = \frac{\frac{i}{vq_H + i(1-q_H)}}{n^*_H + \frac{1-h}{h} n^*_L + \frac{i}{v}\left[\frac{v + i(1-q_H)}{vq_H + i(1-q_H)} + \frac{v + i(1-q_L)}{vq_L + i(1-q_L)} \frac{1-h}{h}\right]},$$

$$\Omega^H_{n^*_H+1} = \frac{i(1-q_H)}{\nu}\frac{\frac{i}{vq_H+i(1-q_H)}}{n^*_H + \frac{1-h}{h}n^*_L + \frac{i}{v}[\frac{v+i(1-q_H)}{vq_H+i(1-q_H)} + \frac{v+i(1-q_L)}{vq_L+i(1-q_L)}\frac{1-h}{h}]}$$

$$\Omega^L_n = \frac{\frac{1-h}{h}}{n^*_H + \frac{1-h}{h}n^*_L + \frac{i}{v}[\frac{v+i(1-q_H)}{vq_H+i(1-q_H)} + \frac{v+i(1-q_L)}{vq_L+i(1-q_L)}\frac{1-h}{h}]} \ \textit{for}\ n = 0, 1, ..., n^*_L - 1,$$

$$\Omega^L_{n^*_L} = \frac{\frac{i}{vq_L+i(1-q_L)}\frac{1-h}{h}}{n^*_H + \frac{1-h}{h}n^*_L + \frac{i}{v}[\frac{v+i(1-q_H)}{vq_H+i(1-q_H)} + \frac{v+i(1-q_L)}{vq_L+i(1-q_L)}\frac{1-h}{h}]},$$

$$\Omega^L_{n^*_L+1} = \frac{i(1-q_L)}{\nu}\frac{\frac{i}{vq_L+i(1-q_L)}\frac{1-h}{h}}{n^*_H + \frac{1-h}{h}n^*_L + \frac{i}{v}[\frac{v+i(1-q_H)}{vq_H+i(1-q_H)} + \frac{v+i(1-q_L)}{vq_L+i(1-q_L)}\frac{1-h}{h}]}$$

Hence the average vintage of the input in the common market follows as:

$$\mathbb{E}_{\mu^L_{n_j}}[a^{n_j}] = \frac{1}{1 - q_L\Omega^L_{n^*_L} - q_H\Omega^H_{n^*_H} - \Omega^L_{n^*_L+1} - \Omega^H_{n^*_H+1}} \cdot \left[\sum_{n=0}^{n^*_H-1}\Omega^H_n a^n + \sum_{n=0}^{n^*_L-1}\Omega^L_n a^n + (1-q_L)\Omega^L_{n^*_L}a^{n^*_L} + (1-q_H)\Omega^H_{n^*_H}a^{n^*_H}\right],$$

Again, note that when $q_L = q_H = 1$, the average vintage of input is the same as shown in Section 4.9. Furthermore, when $q_L = q_H = 0$, the average vintage of input is the one with flipping point at $\{n^*_L + 1, n^*_H + 1\}$.
Remark: $\mathbb{E}_{\mu^L_{n_j}}[a^{n_j}]$ *is continuously increasing in* (q_L, q_H).
Given the property, lets revisit the equilibrium condition for product lines' innovation policies:

$$n^*_L = \min\left\{n \in \mathbb{Z}\cup\{0\} : i[a^nC_L + B_L(n+1) - B_L(n)] + m^L_b D_L \le \nu[hB_H(0) + (1-h)B_L(0) - B_L(n)]\right\}.$$

$$n^*_H = \min\left\{n \in \mathbb{Z}\cup\{0\} : i[a^nC_L + B_H(n+1) - B_H(n)] + m^L_b D_L + m^H_b D_H \le \nu[hB_H(0) + (1-h)B_L(0) - B_H(n)]\right\}.$$

and, there exists $\{q^L_{n^*_L}, q^H_{n^*_H}\} \in [0,1]\times[0,1]$ such that:

$$i[a^{n^*_L}C_L + B_L(n^*_L+1) - B_L(n^*_L)] + m^L_b D_L(q^L_{n^*_L}, q^H_{n^*_H}) \ge \nu[hB_H(0) + (1-h)B_L(0) - B_L(n^*_L)]$$

$$i[a^{n^*_H}C_L + B_L(n^*_H+1) - B_H(n^*_L)] + m^L_b D_L(q^L_{n^*_L}, q^H_{n^*_H}) + m^H_b D_H(q^H_{n^*_H}) \ge \nu[hB_H(0) + (1-h)B_L(0) - B_H(n^*_H)]$$

and equality holds if $q^L_{n^*_L} + q^H_{n^*_H} > 0$. This construction ensures the existence of fixed point by allowing convex policy space with $\{q^L_{n^*_L}, q^H_{n^*_H}\} \in [0,1]\times[0,1]$. That is, there always exists a pair $q^L_{n^*_L} + q^H_{n^*_H} \ge 0$ such that, the solution to Lemma 2 and 3 by taking

$\{\mathbb{E}_{\mu^{\phi}_{n_j}}[a^{n_j}]\}_{\phi\in\{L,H\}}$ as given, n^*_L, n^*_H, coincides with $\{\mathbb{E}_{\mu^{\phi}_{n_j}}[a^{n_j}]\}_{\phi\in\{L,H\}}$, solving the fixed-point problem. To sum up, we refine the definition of the stationary equilibrium:

Definition 1^* ***(Stationary Equilibrium with Perfect Foresight):*** *A stationary equilibrium of this economy is a tuple:*

$$\left\{\left\{\{x^{\phi}_b, x^{\phi}_s\}, \{n^*_{\phi}\}, \{q^{\phi}_{n_\phi}\}, \{m^{\phi}_a, m^{\phi}_b\}, P_{\phi}, T_{\phi}, \{\Omega^{\phi}_n\}^{n^*_{\phi}}_{n=0}, \mathbb{E}_{\mu^{\phi}_{n_j}}[a^{n_j}], V_{\phi}\right\}_{\phi\in\{L,H\}}, g, r\right\}$$

such that:

(1) $\{x^{\phi}_b, x^{\phi}_s\}_{\phi\in\{L,H\}}$*, the buying and selling threshold for inputs maximizes the value of product lines ; (2)* $\{n^*_{\phi}\}_{\phi\in\{L,H\}}$ *are the optimal innovation policies solved in Lemma 2 & 3; (3)* $\{m^{\phi}_a, m^{\phi}_b\}_{\phi\in\{L,H\}}$ *are the input market tightness; (4)* $\{P_{\phi}\}_{\phi\in\{L,H\}}$ *are the pricing policy of input under Nash-bargaining; (5)* $\{T_{\phi}\}_{\phi\in\{L,H\}}$ *is the sales agent's problem stated in Section 4.8; (6) the stationary equilibrium distributions of incremental innovation index:* $\{\{\Omega^{\phi}_n\}^{n^*_{\phi}+1}_{n=0}\}_{\phi\in\{L,H\}}$*; (7) the average vintage of input market at the stationary equilibrium:* $\{\mathbb{E}_{\mu^{\phi}_{n_j}}[a^{n_j}]\}_{\phi\in\{L,H\}}$*, (8) and its associated indifferent product lines between* n^*_{ϕ} *and* $n^*_{\phi}+1$ *fraction:* $\{q^{\phi}_{n_\phi}\}_{\phi\in\{L,H\}}$*, and (9) the value functions of product lines* $\{V_{\phi}\}_{\phi\in\{L,H\}}$*; and (9) the growth rate* g*.*

References

[1] Acemoglu, Daron, and Dan Cao. Innovation by entrants and incumbents. Journal of Economic Theory, 157 (2015): 255-294.

[2] Acemoglu, Daron, Ufuk Akcigit, Harun Alp, Nicholas Bloom, and William Kerr. Innovation, Reallocation, and Growth. American Economic Review, 108.11 (2018): 3450-3491.

[3] Aghion, Philippe, and Peter Howitt. A Model of Growth through Creative Destruction. Econometrica, 60.2 (1992): 323-51.

[4] Akcigit, Ufuk, and William Kerr. Growth through Heterogeneous Innovations.Journal of Political Economy, 126.4 (2018): 1375-1443.

[5] Akcigit Ufuk, Murat Alp Celik, and Jeremy Greenwood. Buy, Keep, or Sell: Economic Growth and the Market for Ideas. Econometrica, 84.3 (2016): 943-84.

[6] Armstrong, Mark, and Jidong Zhou. *Search Deterrence.* The Review of Economic Studies, 83.1 (2016): 26-57.

[7] Bai, Yan, Jose-Vctor Ros-Rull, and Kjetil Storesletten. *Demand Shocks that Look Like Productivity Shocks.* Working Paper, (2012).

[8] Baqaee, David, and Emmanuel Farhi. Productivity and Misallocation in General Equilibrium. Working Paper, (2017).

[9] Bartelme, Dominick, and Yuriy Gorodnichenko. *Linkages and Economic Development.* NBER Working Paper, (2014).

[10] Bas, Maria, and Vanessa Strauss-Kahn. Input-trade liberalization, export prices and quality upgrading. Journal of International Economics, 95.2 (2015): 250-62.

[11] Bena, Jan, and Kai Li. Corporate Innovations and Mergers and Acquisitions. The Journal of Finance, 69.5 (2014): 1924-60.

[12] Bigio, Saki, and Jennifer LaO. *Distortions in Production Networks.* NBER Working Paper, (2016).

[13] Boehm, Johannes, and Ezra Oberfield. Misallocation in the Market for Inputs: Enforcement and the Organization of Production. NBER Working Paper, (2019).

[14] Brynjolfsson, Eric, Daniel Rock and Chad Syverson. The Productivity J-Curve: How Intangibles Complement General Purpose Technologies. NBER Working Paper, (2019).

[15] Buera, Francisco, J., Joseph P. Kaboski, and Yongseok Shin. Finance and Development: A Tale of Two Sectors. The American Economic Review, 101.5 (2011): 1964-2002.

[16] Busse, Meghan, Jorge Silva-Risso, and Florian Zettelmeyer. *$1,000 Cash Back: The Pass-Through of Auto Manufacturer Promotions.* American Economic Review, 94.4 (2006): 1253-70.

[17] Caliendo, Lorenzo, Fernando Parro, and Aleh Tsyvinski. *Distortions and the Structure of the World Economy.* Working Paper, (2017).

[18] Caprettini, Bruno, and Antonio Ciccone. *Turnover Taxes and Productivity Evidence from a Brazilian Tax Reform.* Working Paper, (2015).

[19] Chiu, Jonathan, Cesaire Meh, and Randall Wright. Innovation and Growth with Financial, and other, Frictions . International Economic Review, 58.1 (2017): 95-125.

[20] Dafny, Leemore, S. *Are Health Insurance Markets Competitive?* American Economic Review, 100.4 (2010): 1399-1431.

[21] David, Joel, M, Hugo Hopenhayn, and Venky Venkateswaran. Information, Misallocation and Aggregate Productivity. Quarterly Journal of Economics, 131 (2016): 943-1005.

[22] Diamond, Peter, A. Wage Determination and Efficiency in Search Equilibrium. The Review of Economic Studies, 49.2 (1982): 217-227.

[23] Fadinger, Harald, Christian Ghiglino, and Mariya Teteryatnikova. *Income Differences and Input-Output Structure.* CEPR Discussion Working Paper, (2016).

[24] Garcia-Macia, Daniel, Chang-Tai Hsieh and Peter J. Klenow. How Destructive is Innovation? Econometrica, Forthcoming, (2019).

[25] Gavazza, Alessandro. An Empirical Equilibrium Model of a Decentralized Asset Market. Econometrica, 84.5 (2016): 1755-98.

[26] Goldberg, Pinelopi Koujianou. Dealer Price Discrimination in New Car Purchases: Evidence from the Consumer Expenditure Survey. Journal of Political Economy, 104.3 (1996): 622-654.

[27] Goldberg, Pinelopi Koujianou, Amit Khandelwal, Nina Pavcnik, and Petia Topalova. Imported Intermediate Inputs and Domestic Product Growth: Evidence from India. The Quarterly Journal of Economics, 125.4 (2010): 1727-67.

[28] Hsieh Chang-Tai, and Peter Klenow. Misallocation and Manufacturing TFP in China and India. The Quarterly Journal of Economics, 124.4 (2009): 1403-1448.

[29] Jansen, Justin, J.P., Frans A. J. Van Den Bosch, and H.W. Volberda. Exploratory Innovation, Exploitative Innovation, and Performance: Effects of Organizational Antecedents and Environmental Moderators. Management Science, 52.11 (2006): 1661-74.

[30] Jones, Charles I. *Misallocation, economic growth, and input-output.* In Acemoglu, D., Arellano, M., and Dekel, E., editors, Advances in Economics and Econometrics, Vol. II, Cambridge University Press economics (2013): 419-454.

[31] Joskow, Paul, L. Contract Duration and Relationship-Specific Investments: Empirical Evidence from Coal Markets. The American Economic Review, 77.1 (1987): 168-85.

[32] Khandelwal, Amit, and Petia Topalova. Trade Liberalization and Firm Productivity: The Case of India. The Review of Economics and Statistics, 93.3 (2011): 995-1009.

[33] Klette, Jakob, and Samuel Kortum. Innovating Firms and Aggregate Innovation. Journal of Political Economy, 112.5 (2004): 986-1018.

[34] Lagos, Ricardo, Guillaume Rocheteau, and Randall Wright. *Liquidity: A New Monetarist Perspective.* Journal Of Economic Literature, 55.2 (2017): 371-440.

[35] Lentz, Rasmus, and Dale T. Mortensen. An Empirical Model of Growth Through Product Innovation. Econometrica, 76.6 (2008): 1317-73.

[36] McCall, J.J. Economics of Information and Job Search. The Quarterly Journal of Economics, 84.1 (1970): 113-126.

[37] Midrigan, Virgiliu, and Daniel Yi Xu. Finance and Misallocation: Evidence from Plant-Level Data. The American Economic Review, 104.2 (2013): 442-58.

[38] Mortensen, Dale. The Matching Process as a Noncooperative Bargaining Game. The Economics of Information and Uncertainty, (1982): 223-58.

[39] Osotimehin, Sophie, and Latchezar Popov. *Misallocation and intersectoral linkages.* Working Paper, (2017).

[40] Ottonello, Pablo. *Capital Unemployment.* Working Paper, (2019).

[41] Pavcnik, Nina. *Trade Liberalization, Exit, and Productivity Improvements: Evidence from Chilean Plants.* The Review of Economic Studies,69.1 (2002): 245-276.

[42] Phillips Gordon, M., and Alexei Zhdanov. RD and the Incentives from Merger and Acquisition Activity. The Review of Financial Studies, 26.1 (2013): 34-78.

[43] Piazzesi, Monika, Martin Schneider and Johannes Stroebel. *Segmented Housing Search.* NBER Working Paper, (2013).

[44] Pissarides, Chris. Equilibrium Unemployment Theory, (1990).

[45] Restuccia, Diego, and Richard Rogerson. Policy distortions and aggregate productivity with heterogeneous establishments. Review of Economic Dynamics, 11.4 (2008): 707-20.

[46] Romer, Paul, M. Endogenous Technological Change. Journal of Political Economy, 98.5 (1990): 71-102.

[47] Salz, Tobias. *Intermediation and Competition in Search Markets: An Empirical Case Study.* Working Paper, (2015).

[48] Scott-Morton, Fiona, M., Florian Zettelmeyer, and Jorge Silva-Risso. *Internet Car Retailing.* Journal of Industrial Economics 49.4 (2001): 501-19.

[49] Town, Robert, and Gregory Vistnes. *Hospital competition in HMO networks.* Journal of Health Economics, 20.5 (2001): 733-53.